RBEE

Stability in matching models with capacity constraint with agents' responsive preferences, the set of q_E-stable-G

Abstract: *In this paper a variant to the many-to-one matching model is presented, in which two types of complementary agents and an institution intervene. The latter wants to assign agents to perform certain tasks, each of which can be done by one agent from one set with many agents from the other. The institution has preferences over the possible matchings and a quota q, which is the maximum number of agents it can hire. In this model, considering responsive references for the agents, two concepts of stability are extended in a natural way and the concepts of q-stability-R and q-stability-G are defined. It is shown, under the institution's responsive preference costraint, that there are sets of the matchings q-stable-G, and their complete characterization is obtained.*

Keywords: *Matching; Quota; Stability; Restriction.*

Classificação JEL: C78; C71; D79.

Delfina Femenia[1]

Alicia Gimenez[2]

Mario Videla[3]

Flavia Zalazar[4]

[1] Universidad Nacional de San Juan
E-mail: delfinafemenia@gmail.com.

[2] Universidad Nacional de San Juan
E-mail: alifangi@gmail.com.

[3] Universidad Nacional de San Juan
E-mail: maredvid@gmail.com.

[4] Universidad Nacional de San Juan
E-mail: lic.flaviazalazar@gmail.com.

1. Introduction

Many-to-one assignment models are used to study market problems whose distinctive feature is that the agents involved from the outset are in disjoint sets with different characteristics (e.g., principals and students, firms and workers, etc.). The nature of the problem studied here consists of assigning to an agent from one set many agents from the other. In the case in which to an agent in a set one agent is assigned at the most, the model is called one-to-one matching model. Agents not matched to any agent in the other set are called *singles*.

The "College admissions problem" is the name given by Gale and Shapley, in 1962, to the simplest of the many-to-one models. They assume that companies have a maximum number of positions to cover (the quota), and that each company has a preference relationship in the different groups of workers, and each worker has a preference relationship in the whole of the companies. A central theme in formulating a many-to-one model is how to model the preferences of firms as this involves comparison of different groups of workers. The simplest preference of acceptance of the firms is the *responsive* preference. The company, before any pair of subsets of workers which differ in only one worker, prefers the subset containing the most preferred worker according to their individual preference over each worker. Given a preference profile, an assignment or matching satisfies a specific property of stability if it cannot be blocked, in a sense to be specified below, by any agent or any unassigned pair of agents.

Roth and Sotomayor, in 1990, studied the most general of many- to- one models, which they called "College admissions problem with substitutable preferences". Each company has a substitutive preference relationship over different groups of workers, i.e. each company prefers to hire a worker, even if other workers are no longer available, regardless of their individual preferences over each worker. In 1982, Kelso and Crawford were the first to use this property in the most general model several-to-one with money.

A variant to the one-to-one matching model is the *one-to-one matching model with capacity constraint*, presented by Femenia, Marí, Neme and Oviedo in 2011. They assume that two sets of complementary agents and one institution intervene and the model consists of assigning each worker on one side of the market to a worker on the other side, such that the pairs of workers hired by the institution are q, at most. That is to say, the institution will have to choose q pairs of workers at most in agreement with its order of preference. The stability property in this model depends on the preferences expressed by the participants and on the institution's preferences; this is why the property of q-stability is defined. Under the constraint of the institution's *responsive preferences*, the existence of the set of q-stable matchings is guaranteed and it is possible to obtain its characterization. Femenia y Marí, in 2012, presented an application of these results to the real estate market with the variant that in the matching process the state intervene. A variant is proposed to the several-to-one model, under the constraint of the institution's *responsive preferences* with substitutable preferences for one side of agents, presented by Femenia y Marí, in 2020.

In this work, a variant is proposed to the several-to-one model. It is assumed that an institution U wants to assign each agent in a set D to several agents in a set E. The institution has preferences over the agents of both sets and a quota q_E, which is the maximum number of agents it can assign. In addition, it is natural to think that there may be more candidates than positions to be filled. For example, a university has members on its staff who are prospective scholarship directors and gives a certain number of grants to be distributed among students in an optimum way.

The stability property in this model depends on the participants and the institution's preferences. As in several-to-one matching models, the case is considered in which the directors have responsive preferences.

This work is organized as follows. Section 2 presents a brief review of the theoretical concepts of several-to-one, one-to-one and one-to-one matching with capacity constraints models. The most important definitions and the results that guarantee the existence of stability in these models are included. Section 3 describes the several-to-one matching model with capacity constraints. For directors' responsive preferences the *q*-blocking concept of the one-to-one model with capacity constraint is extended in a natural way to this new model. Considering two criteria for such natural extension, the concepts of q_E-G-blocking and q_E-blocking-R are defined. Under the institution's responsive preference constraints, the existence of the set of q_E-stable-R matchings is shown. Some of the demonstrations set forth in this chapter are developed in the appendix. Finally, in section 4, under the institution's responsive preference constraint, a characterization of the set of q_E-stable-R is obtained.

2. Preliminaries: Bilateral Matching Models

2.1 Many-to-one matching models

The many-to-one matching model consists of two disjoint sets $D = \{d_1,...,d_n\}$ and $E = \{e_1,...,e_n\}$. Each agent $e \in E$ has a strict, complete and transitive preference relation P_e over set $D \cup \{\varnothing\}$, and each agent $d \in D$ has a strict, complete and transitive preference relation P_d over set 2^E. These preferences determine that an agent *f* of set $E(o\,D)$ prefers alternative *a* to alternative *b*, where *a* and *b* are agents of set *D* (or subsets of set *E*), if and only if *a* precedes *b* in the list of preferences of *f*. A preference profile is (*n+m*)-uplas preference relations of the agents of set *D* and the agents of set *E* and is represented by **P** = $(P_{d1},...,P_{dn};P_{e1},...,P_{en})=(P_D,P_E)$. Given a preference P_e, the agents of *D* preferred by *e* to set $\varnothing$ are acceptable for *e*. Similarly, given a preference P_d, the subsets of *E* preferred by *d* over to set $\varnothing$ are acceptable for *d*.

In order to describe the preferences of an agent, we adopt an abbreviated list that includes only the agents or subsets acceptable for it. For example,

$P_{di}=\{e_1,e_3\},\{e_2\},\{e_1\}$ $\quad P_{ej}=d_1,d_3$

Given two disjoint sets *D* and *E*, and a preference profile **P**, the many-to-one matching model will be denoted by:

$$\bar{M} = (D, E, \mathbf{P})$$

A solution of the many-to-one mapping model is a mapping that assigns a subset of agents of *E* to each agent of *D*. Formally:

Definition 2.1 An *assignment* or *matching* is a function $\mu : D \cup E \rightarrow 2^{D \cup E}$ such that, for every $d \in D$ and $e \in E$, it satisfies:

1. $\mu(e) \subseteq D$ and $\#\mu(e) = 1$ or else $\mu(e) = \varnothing$
2. $\mu(d) \in 2^E$
3. $\mu(e) = \{d\}$ if and only if $e \in \mu(d)$

Note 2.1 In condition 3, for language abuse we will use, *μ(e) = d* will be used instead of *μ(e) = {d}.*

The set of all the possible matchings in model $\bar{M}$ will be denoted by $\bar{\boldsymbol{M}}$.

Given a model $\bar{M} = (D, E, \mathbf{P})$ and a matching $\mu \in \bar{\boldsymbol{M}}$ the following subsets of *D* and *E* will be considered, respectively:

$$\mu(E)=\{d\in D:\mu(d)\neq\emptyset\}\ \text{ and }\ \mu(D)=\{e\in E:\mu(e)\neq\emptyset\}$$

As usual, the cardinality of sets $\mu(E)$ and $\mu(D)$ is denoted by $\#\mu(E)$ and $\#\mu(D)$, respectively. Set $\mu(E)$ is a set of agents of set D, which suggests that $\#\mu(E)$ be denoted as $\#_D\mu$ and, similarly, $\#\mu(D)$ be denoted by $\#_E\mu$; that is to say, $\#_E\mu = \#\mu(D)$ and $\#_D\mu = \#\mu(E)$.

The subset of E most preferred by d with respect to the preference P_d is called choice set of agents of E with respect to the preferences of agent d and denoted by $Ch(E',P_d)$.

$$Ch(E',P_d)=\max_{P_d}\{E''\subseteq E':E''P_dE'\}$$

A matching $\mu\in\bar{M}$ is **individually rational** if it is not blocked by any agent.

A matching $\mu\in\bar{M}$ is **blocked by a pair** *(d,e)* if $e\notin\mu(d), dP_e\mu(e)$ and $e\in Ch(\mu(d)\cup\{e\},P_d)$.

A matching $\mu\in\bar{M}$ is **stable** *if it is not blocked by an agent or by a pair of agents*.

Given a many-to-one matching model $\bar{M}=(D,E,\mathbf{P})$, the set of stable matchings in $\bar{M}$ is denoted by $S(\bar{M})$.

A matching μ is said to be one-to-one if each agent d is assigned to an agent e, i.e. Condition 2 of Definition 2.1 is replaced by: $\mu(d)\subseteq E$ and $\#\mu(d) = 1$ or else $\mu(d) = \emptyset$. The model in which every matching is one-to-one is known in the literature as the *marriage model* or *one-to-one matching model*. Such a model is denoted by $\bar{M}=(D,E,\mathbf{P})$, $\mathrm{P}=(P_d, P_e)$, P_d and P_e being preferences over sets $E\cup\{\emptyset\}$ and $D\cup\{\emptyset\}$, respectively. ***M*** is the set of all possible matchings in the model and $S(M)$, the set of stable matchings.

Theorem 2.1 (*Gale and Shapley, 1962*) *If* $\bar{M}=(D,E,\mathbf{P})$ *is a one-to-one matching model, then* $S(M)\neq\emptyset$.

It is known that in the many-to-one model, set $S(\bar{M})$ may be empty. (See example 2.7 in *Two-sided matching: a study in game-theoretic and analysis* [11]). Such a result is the reason why the literature has focused on the constraint of the preferences of agents d.

Let us consider that for each agent d there is a positive integer q_d called the quota of the agent d and let us consider the most general many-to-one model $\bar{M}=(D,E,\mathbf{P})$, in which the agents $d\in D$ consider each agent $e\in E$ one another's responsive.

Definition 2.2 *The preference relation P_d is* ***q_d-responsive*** *with respect to the individual preferences P_d over $E\cup\{\emptyset\}$ if*

- For all $E'\subseteq E$ such that $\#E' < q_d$ and $e\notin E'$ verifies $E'\cup\{e\}P_dE'$ if and only if $eP_d\emptyset$.
- $\emptyset P_dE'$, for every E' such that $\#E' > q_d$.
- For every E', $e'\in E'$ and $e'\notin E'$, it is verified that $(E'\setminus\{e'\})\cup\{e\}P_dE'$ if and only if eP_de'.

Note 2.2 To the end of simplifying notation, the expression $(E'\setminus\{e'\})\cup\{e\}$ is written $E'\setminus e'\cup e$.

Remark 2.1 *Taking into account the definition of choice, we can state that for q_d-responsive preferences, the following is verified:*

- If $eP_d\emptyset$, for each $e\in E'$ and $\#E'\leq q_d$, then $E' = Ch(E',P_d)$.
- If $e\notin E'$ and eP_de', for some $e'\in E'$, then $e\in Ch(E'\cup e,P_d)$.

From now onwards, in order to refer to the many-to-one model $\bar{M} = (D, E, \mathbf{P})$ with q_d-responsive preferences, either of these notations are possible: $\bar{M} = (D, E, \boldsymbol{P}, q_d)$ or $\bar{M} = (D, E, \mathbf{P})$, with D-responsive preferences.

If for each $d \in D$, $q_d = 1$, the several-to-one model $\bar{M} = (D, E, \boldsymbol{P}, q_d)$ is reduced to the one-to-one $\bar{M} = (D, E, \mathbf{P})$ and for all $\mu \in \bar{\boldsymbol{M}}$, is verified.

$$\#\mu = \#\mu_E = \#\mu_D.$$

Some valid results, necessary for subsequent sections, are presented.

Theorem 2.2 (*Gale and Shapley, 1962*) *If* $\bar{M} = (D, E, \boldsymbol{P}, q_d)$ *is a many-to-one model, then model, then* $S(\bar{M}) = \varnothing$.

Theorem 2.3 (*Roth and Sotomayor, 1990) Let* $\bar{M} = (D, E, \boldsymbol{P}, q_d)$ *be a many-to-one model and* $\mu \in S(M)$*, for every* $d \in D$ *the following is verified:*

- If $\mu(d) = \varnothing$, then $\bar{\mu}(d) = \varnothing$, for all $\bar{\mu} \in S(\bar{M})$.
- $\# \mu(d) = \# \bar{\mu}(d)$, for all $\bar{\mu} \in S(\bar{M})$.

For agents $e \in E$, Theorem 2.3 is stated in a symmetrical way.

Remark 2.2 Theorem 2.3 indicates that, for a many-to-one model $\bar{M} = (D, E, \boldsymbol{P}, q_d)$, the set of agents not assigned is the same for every stable matching; therefore, the cardinality of any stable matching in $\bar{M} = (D, E, \boldsymbol{P}, q_d)$ is the same, i.e. si $\mu \in S(\bar{M})$ and $\mu' \in S(\bar{M})$, then $\#_E\mu = \#_E\mu'$ and $\#_D\mu = \#_D\mu'$.

Theorem 2.4 (*Roth, 1986) Let* $\bar{M} = (D, E, \boldsymbol{P}, q_d)$ *be a many-to-one model and* $\mu \in S(\bar{M})$*, for every* $d \in D$*, the following is verified:*

$$\textit{If}\ \# \mu(d) < q_d, \text{then } \mu(d) = \bar{\mu}(d), \text{ for all } \bar{\mu} \in S(\bar{M})$$

The lemma that follows, which will be useful later, shows a relation between the cardinality of a stable matching of model $\bar{M}$ and the cardinality of a stable matching of the reduced model $\bar{M}'$.

Lemma 2.1 (Femenia, Marí, Neme and Oviedo in 2011) *Given the several-to-one models* $\bar{M} = (D, E, \boldsymbol{P}, q_d)$ *and* $\bar{M}_{E'} = (D, E', \boldsymbol{P'}, q_d)$*, with* $E' \subseteq E$ and $\boldsymbol{P}' = \{P_D', P_{E'}\}$*, if* $\mu \in S(\bar{M})$ *and* $\mu' \in S(M_{E'})$*, then* $\#_E \mu' \leq \#_E\mu \leq \#_E\mu' + \#(E \backslash E')$.

2.2 One-to-one matching model with capacity restriction

In 2011, Femenia, Marí, Neme and Oviedo [3] presented a variant to the one-to-one matching model in which two sets of complementary agents and an institution are involved. The institution wants to assign agents to do certain tasks which can be carried out by a pair of complementary agents. It has preferences over each of the pairs of agents it can assign. Many times, the institution has a quota q, which is the maximum number of pairs of agents it can assign.

It must be noted that, even though in this model two sets of workers and an institution are involved, this model is not equivalent to the trilateral matching model introduced by Alkan [1] in 1986, in which there is no stability.

This model consists of two finite and disjoint sets of agents $D = \{d_1,...,d_n\}$ and $E = \{e_1,...,e_m\}$, respectively. Each worker $d \in D$ has a strict preference relation[1] P_d over the set of agents 2^E and each worker $e \in E$ has a strict preference relation P_e over the set of

[1] A preference is a binary, reflexive, antisymmetric, transitive and complete relation.

agents $D \cup \{\varnothing\}$.

Preference profiles are *(n+m)*-tuples of preference relations represented by $P = (P_{d1},...,P_{dn}) ; P_{e1},...,P_{em}) = (P_D, P_E)$, and an institution denoted by *U*. Institution *U* has a binary relation R_U over the set of all possible matchings ***M***, the empty matching included. Let P_U and I_U denote the strict and indifferent preference relations induced by R_U, respectively. Suppose now that the institution can assign a maximum number of positions - quota $q \leq \min\{n, m\}$- to be filled; then, only the matchings whose cardinality is smaller or equal to *q* may be acceptable. The institution may choose some matchings of *M* according to its preference P_U and their quota restriction *q*. We denote $\boldsymbol{M}_q = \{\mu \in \boldsymbol{M} : \# \mu \leq q\}$.

This new matching marker is denoted by $M_U^q = (M, R_U, q)$. A matching μ is *acceptable* for institution *U* according to their preferences if $\mu \in \boldsymbol{M}_q$ and $\mu R_U \mu^{\varnothing}$, in which $\mu^{\varnothing}$ is the matching such that $\mu^{\varnothing}(x) = \varnothing$, for every $x \in D \cup E$. Given *M* and a quota $q \leq \min\{n, m\}$, the institution can only accept assignments of *M* which are most preferred to the empty matching according to its preference P_U, and its cardinal is not larger than the allowed number of positions $\#\mu \leq q$. A matching is acceptable if the partner assigned is preferred to the empty set. Formally,

Definition 2.3 Given a model M_U^q, an assignment μ is ***q**-individually rational* if $\#\mu \leq q$, $\mu P_U \mu^{\varnothing}$ and for every $f \in D \cup E$ such that $\mu(f) \neq \varnothing, \mu(f) P_f \varnothing$ is verified.

Given an assignment $\mu \in \boldsymbol{M}$ and a pair of workers $(d,e) \in D \times E, \mu_{(d,e)}$ is defined as follows:

$$\mu_{(d,e)}(f) = \begin{cases} \mu(f) \ \ if \ f \notin \{d, e, \mu(d), \mu(e)\} \\ d \quad if \qquad f = e \\ \varnothing \qquad\qquad otherwise. \end{cases}$$

Notice that if $\mu(d) = e$, then $\mu_{(d,e)} = \mu$.

Note 2.3 The matching $\mu_{(d,e)}$ may not be individually rational. Let us consider a matching μ such that $\#\mu = q$ and let *(d,e)* such that, if $\mu(d) = \varnothing = \mu(e)$, then $\#\mu_{(d,e)} > q$ and $\mu_{(d,e)}$ is not *q*-individually rational.

Usually, in standard models, *(d,e)* is a blocking pair if these agents are not assigned to each other and if they each other to their current partners. Note that in our model, we may have a blocking pair *(d,e)* such that the matching that the blocking pair satisfies is not acceptable for institution *U*. Will consider two types of blocking pairs for μ. One type is that which occurs when the assignment μis blocked by a couple of agents in the institution, already assigned by the matching, and the other is the type in which the assignment is blocked by a pair of agents, one of whom at least is single. In this case, the assignment obtained, which satisfies the blocking pair, is preferred by the institution to the assignment μ. Formally:

Definition 2.4 A matching μ is *q-blocked* by a pair of workers *(d,e)* if

1. $eP_d \mu(d), dP_e \mu(e),$ and
2. either
   (a) $\mu(d) \in E$ and $\mu(e) \in D$, or
   (b) $\mu_{(d,e)}$ is q-individually rational and $\mu_{(d,e)} R_U \mu$,

Definition 2.5 *A matching μ is q-stable if it is q-individually rational and is not q-blocked by any pair of workers.*

Given a matching market $M_U^q = (M, R_U, q), S(M_U^q)$ denotes the set of *q*-stable matchings. Notice that in Femenia, Marí, Neme and Oviedo 2011 [3], it was proved that, under the restriction of the institution's responsive preferences, the set of *q*-stable matchingsis non-empty, i.e. $S(M_U^q) \neq \varnothing$. They also obtained a characterization of this set as: $S(M_U^q) = T_q(M) \cup T_{<q}(M)$.

Note 2.4 The definition of the institution´s responsive preferences and of sets $T_q(M)$ and $T_{<q}(M)$ are given in detail in Appendix.

3. Characterization The model

A variant to the many-to-one matching model will be considered now in which two sets of complementary agents and an institution are involved. The institution wants to assign agents to do certain tasks each of which can be performed by one agent of a set of many agents of the complementary set. The institution has preferences over each of the pairs of agents it can assign. Unlike the one-to-one matching model with capacity restriction this model matches an agent from set *D* with many agents from set *E*, and the institution has a quota which is the maximum number of agents from *E* it can assign. Since the institution's quota limitation is given over agents from set *E*, it is symbolized with q_E. Each $d \in D$ has a maximum number of agents from *E* with which it may be designated which will be indicated with q_d. We assume that $q_E \leq \min\{\#E, \sum_{d \in D} q_d\}$.

This new matching model is called many-to-one matching model with capacity restriction and denoted by $\bar{M}_U^{q_E} = (\bar{M}, R_U, q_E)$. The set of all matchings in this model is symbolized with $\bar{\boldsymbol{M}}_{q_E}$, i.e., $\bar{\boldsymbol{M}}_{q_E} = \{\mu \in \bar{\boldsymbol{M}} : \#_E \, \mu \leq q_E\}$, where $\#_E \, \mu = \#\{e \in E : \mu(e) \neq \varnothing\} = \sum_{e \in E} \#\mu(e)$. The notion of the q_E-individually rational matching of the one-to-one model with capacity restriction is extended naturally to the many-to-one model with capacity restriction as follows.

Definition 3.1 Given a model $\bar{M}_U^{q_E}$, a matching $\mu \in \bar{M}_U^{q_E}$ is q_E-individually rational if for all $e \in E$, $\mu(e) P_e \varnothing$, for all $d \in D$, $\mu(d) = Ch(\mu(d), q_d, P_d)$ and $\mu \mathrm{R}_U \mu^{\varnothing}$.

Consider the model $\bar{M}_U^{q_E} = (\bar{M}, R_U, q_E)$ with D-responsive preferences. It is noted that if for every $d \in D$, $q_d = 1$, the model is reduced to the one-to-one model with capacity restriction. The objective is to extend naturally to this model the definition of *q*-blocking of the one-to-one model with capacity restriction.

Note that considering *μ(d)* ≠ ∅, in the one-to-one assignment model is equivalent to considering *#μ(d) = 1 =* q_d, however, in the many-to-one assignment model this equivalence is not always is given in the case where *#μ(d) <* q_d. This suggests that we consider two criteria to extend the notion of q_E- blocking in this model.In order to formalize both criteria we define the following matching:

Given a matching $\mu \in \bar{\boldsymbol{M}}$, a pair $(d,e) \in D \times E$ and $E' \subseteq \mu(d) \cup e$,

$$\mu_{(d,e)}^{E'}(f) = \begin{cases} \mu(f) & \text{if } f \notin \{d, e, \mu(e)\} \cup \mu(d) \\ Ch(E', P_d) & \text{if } f = d \\ d & \text{if } f = e \\ \mu(f) \setminus \{e\} & \text{if } f = \mu(e) \\ \varnothing & \text{otherwise} \end{cases}$$

Considering the first criterion. For that let's define first:

Definition 3.2 A matching $\mu \in \bar{M}_{q_E}$ is q_E*-blocked* in sense G by a pair of agents *(d,e)* if

1. $e \notin \mu(d), dP_e\mu(e), e \in Ch(\mu(d) \cup \{e\}, P_d)$, and

2. either

(a) $\mu(d) \neq \emptyset$ and $\mu(e) \neq \emptyset, or$

(b) there exists $E' \subseteq \mu(d) \cup \{e\}$ such that $\mu^{E'}_{(d,e)}$ is q_E-individually rational and $\mu^{E'}_{(d,e)} R_U \mu$.

If a matching μ is q_E*-blocked* in sense *G* by a pair of workers *(d,e)*, we write μ is q_E*- blocked-G* by the pair of agent *(d,e)*.

Definition 3.3 *A matching* $\mu \in \bar{\boldsymbol{M}}_{q_E}$ *is* ***q_E-stable-G*** *if it is q_E-individually rational and is not q_E-blocked-G by any pair of agents*.

Given a matching market $\bar{M}^{q_E}_U = (\bar{M}, R_U, q_E)$, $S_G(\bar{M}^{q_E}_U)$ denotes the set of matchings q_E*-stable-G*. Let us now consider the second criterion, restricting the condition to the case *#μ(d) =* q_d. It is worked on in another paper (Stability in matching models with capacity constraint with agents' responsive preferences, the set of q_E*-stable-R*).

Definition 3.4 A matching $\mu \in \bar{\boldsymbol{M}}_{q_E}$ is q_E*-blocked* in sense R by a pair of agents *(d,e)* if

1. $e \notin \mu(d), dP_e\mu(e), e \in Ch(\mu(d) \cup \{e\}, P_d)$, and

2. either

(a) $\# \mu(d) = q_d$ and $\mu(e) \neq \emptyset$, or

(b) there exists $E' \subseteq \mu(d) \cup \{e\}, \mu^{E'}_{(d,e)}$, is q_E-individually rational and $\mu^{E'}_{(d,e)} R_U \mu$.

If a matching μ is q_E*-blocked* in sense R by a pair of workers *(d,e)*, we write μ is q_E*-blocked-R* by the pair of agent *(d,e)*.

Definition 3.5 *A matching* $\mu \in \bar{\boldsymbol{M}}_{q_E}$ *is* ***q_E-stable-R*** *if it is q_E-individually rational and is not q_E-blocked-R by any pair of agents.*

Given a matching market $\bar{M}^{q_E}_U = (\bar{M}, R_U, q_E)$, $S_G(\bar{M}^{q_E}_U)$ denotes the set of matchings q_E*-stable-R*.

3.1 The institution's responsive preference

A preference $\succ_{2^E}$ of *U* is responsive to $\succ_E$ *such that it satisfies the following condition*, for every *S*, $S' \in 2^E, e \in S$ and $e' \notin S'$ such that $S = S' \setminus \{e\} \cup \{e'\}$, then $S \succ_{2^E} S'$ if , and only if $e' \succ_E e$.

Given a matching market $\bar{M} = (D, E, \mathbf{P})$ and $\mu \in \bar{\boldsymbol{M}}_{q_E}$, in order to formalize the institution's responsive preference, we introduce the notations that follow:

• $B_\mu = \{(d, E') \in D \times 2^E : \mu(d) = E'\}$.

•For every $f \in D \cup E$,

$$\mu^{(d,E')}(f) = \begin{cases} \emptyset \ if \ f \notin \{d\} \cup E' \\ d \ if \quad f \in E' \\ E' if \quad f = d \end{cases}$$

A preference relation R_U is a ***responsive extension** of preferences* $\succ_D$ and $\succ_{2^E}$, such that it satisfies the following conditions:

i) $\mu^{(d,E')} P_U \mu^{\varnothing}$, if and only if $d \succ_D \varnothing, E' \succ_{2^E} \varnothing$

ii) $\mu P_U \mu^{\varnothing}$, if and only if $\mu^{(d,E')} P_U \mu^{\varnothing}$, for every $(d,E') \in B_\mu$

iii) $\mu^{(d,E')} P_U \mu^{(d',E')}$, if and only if $d \succ_d d'$

iv) $\mu^{(d,E')} P_U \mu^{(d,E'')}$, if and only if $E' \succ_{2^E} E''$

v) For every μ, $\mu' \in \bar{\boldsymbol{M}}$ such that $B_{\mu'} \subset B_\mu y\, \mu P_U \mu^{\varnothing}$, then $\mu P_U \mu'$

vi) For every μ, $\mu' \in \bar{\boldsymbol{M}}$, for every $d \in D$ and for every $e \in E$ such that $\#\mu(d) = \#\mu'(d)$ and $\#\mu(e) = \#\mu'(e)$ then $\mu I_U \mu'$

vii) For every μ, $\mu' \in \bar{\boldsymbol{M}}$ such that $\mu(E) = \mu'(E)$, $\mu(D) = \mu'(D) \setminus \{e_1\} \cup \{e_2\}$ and $e_1 \succ_E e_2$, then $\mu' P_U \mu$

viii) For every μ, $\mu' \in \bar{\boldsymbol{M}}$ such that $\mu(D) = \mu'(D)$ for every $d \in D \setminus \{d_1, d_2\}$ such that $\#\mu(d) = \#\mu'(d)$, $\#\mu(d_1) = \#\mu'(d_1)$, $\#\mu'(d_2) = \#\mu(d_2)$, then $d_1 \succ_D d_2$

We consider a preference R_U to be responsive if there are two individual preferences $\succ_D$ and $\succ_{2^E}$ over $D \cup \varnothing$ and 2^E respectively, such that R_U is a responsive extension.

3.2 Existence of q_E-stable-G matchings

For every $t \in N$, we can define the following subset $F^t \subseteq F$ such that $\#F^t = t$, and for every $f \in F^t$ and $f \notin F^t$ we have that $f \succ_F f'$. Note that $F^1 \subseteq F^2 \subseteq \ldots \subseteq F^l = F$, where $\#F = l$.

Given sets $\boldsymbol{d} = \{1,2,\ldots,\#D\}$ and $\boldsymbol{e} = \{1,2,\ldots,\#E\}$, for every $(t_1, t_2) \in \boldsymbol{d} \times \boldsymbol{e}$, we denote $M^{(t1,t2)}$, the restriction of M to D^{t1} and E^{t2}, i.e., $M^{(t1,t2)} = (D^{t1}, E^{t2}, P)$. In the model $\bar{M}_U^{q_E}$, let us now consider only the sets $S(\bar{M}^{(t_1,t_2)})$ whose matchings have cardinality q_E. Given $(t_1, t_2) \in \boldsymbol{d} \times \boldsymbol{e}$, q, and the following sets of matchings:

$$T_q(\bar{M}^{(t_1,t_2)}) = \begin{cases} S(\bar{M}^{(t_1,t_2)}) \ \textit{if}\ \#u = q \text{ for every } \mu \in S(\bar{M}^{(t_1,t_2)}) \\ \varnothing \qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

and $T_q(\bar{M}) = \left\{\mu : \exists\, (t_1,t_2) \text{ such that } \mu \in T_q(\bar{M}^{(t_1,t_2)})\right\}$.

The following proposition gives us some information about the structure of the set $T_q(\bar{M})$, whose proof can be seen in the Appendix.

Proposition 3.1 *Given* $M_U = (M, R_U), (t_1, t_2) \in \boldsymbol{d} \times \boldsymbol{e}$*, there exists* $K \subseteq \boldsymbol{d} \times \boldsymbol{e}$*, such that* $T_{q_E}(\bar{M}) = \dot{\bigcup}_{(t_1,t_2)\in K} T_q(\bar{M}^{(t_1,t_2)})$.

The following lemmas will be used in the proof of the above proposition.

Lemma 3.1 Let $\mu \in T_{q_E}(\bar{M}^{(t_1,t_2)})$ and $(t_1', t_2') \in \boldsymbol{d} \times \boldsymbol{e}$ be such that $t_1' \leq t_1 y\ t_2' \leq t_2$, $\mu(D) \subseteq D^{t_1'}$ and $\mu(E) \subseteq E^{t_2'}$. Then $\mu \in T_{q_E}(\bar{M}^{(t_1',t_2')})$.

Lemma 3.2 Let $(t_1, t_2) \in \boldsymbol{d} \times \boldsymbol{e}$, and $(t_1', t_2') \in \boldsymbol{d} \times \boldsymbol{e}$ be such that $t_1' \leq t_1$ and $t_2' \leq t_2$. Then either $T_{q_E}(\bar{M}^{(t_1,t_2)}) \subseteq T_{q_E}(\bar{M}^{(t_1',t_2')})$ or $T_{q_E}(\bar{M}^{(t_1,t_2)}) \cap T_{q_E}(\bar{M}^{(t_1',t_2')}) = \varnothing$.

Lemma 3.3 Let $(t_1,t_2)\in \boldsymbol{d}\times\boldsymbol{e}$, and $(t_1',t_2')\in \boldsymbol{d}\times\boldsymbol{e}$ be such that $T_{q_E}(\bar{M}^{(t_1,t_2)})\cap T_{q_E}(\bar{M}^{(t_1',t_2')})=\varnothing$. Then exists $(\bar{t}_1,\bar{t}_2)\in \boldsymbol{d}\times\boldsymbol{e}$ such that $T_{q_E}(\bar{M}^{(t_1,t_2)})\subseteq T_{q_E}(\bar{M}^{(\bar{t}_1,\bar{t}_2)})$ and $T_{q_E}(\bar{M}^{(t_1',t_2')})\subseteq T_{q_E}(\bar{M}^{(\bar{t}_1,\bar{t}_2)})$.

The proof of previous lemma are presented in Appendix.

Proposition 3.2 Let $\bar{M}_U^{q_E}=(\bar{M},R_U,q_E)$ be a matching market with quota restriction. Then $T_{q_E}(\bar{M})\subseteq S_G(\bar{M}_U^{q_E})$.

The proof of this proposition is in Appendix.

Given $\bar{M}_U^{q_E}$ and $(t_1,t_2)\in \boldsymbol{d}\times\boldsymbol{e}$, we define the following sets of stable matchings: $T_{<q_E}(\bar{M})=\bigcup_{(t_1,t_2)\in N}T_{<q_E}(\bar{M}^{(t_1,t_2)})$. Being

$T_{<q_E}(\bar{M}^{(t_1,t_2)})=\{S(\bar{M}^{(t_1,t_2)}),\#_E\,\mu<q_E,\ d$ and e are not mutually acceptable, for every $(d,e)\in D\setminus\mu(E^{t_2})\times E\setminus\mu(D^{t_1})$ and $e\notin Ch(\mu(d)\cup e,P_d)$ for every $(d,e)\in$ $\mu(E^{t_2})\times E\setminus\mu(D^{t_1})\}$.

The following proposition gives us some information about set $T_{<q_E}(\bar{M})$, whose proof can be seen in the Appendix.

Proposition 3.3 Let $\bar{M}_U^{q_E}=(\bar{M},R_U,q_E)$ be a matching market with quota restriction and $\hat{K}=\{(t_1,t_2)\in \boldsymbol{d}\times\boldsymbol{e}:(\forall(t_1',t_2')\neq(t_1,t_2),\ t_1'\le t_1,\ t_2'\le t_2:T_{<q_E}(\bar{M}^{(t_1,t_2)})\cap T_{<q_E}(\bar{M}^{(t_1',t_2')})=\varnothing)\}$ then $T_{<q_E}(\bar{M})=\bigcup^{\bullet}_{(t_1,t_2)\in\hat{K}}T_{<q_E}(\bar{M}^{(t_1,t_2)})$.

The following lemmas will be used in the proof of the above proposition.

Lemma 3.4 Let $\mu\in T_{<q_E}(\bar{M}^{(t_1,t_2)})$, $(t_1,t_2)\in \boldsymbol{d}\times\boldsymbol{e}$ and $t_1'\le t_1$ and $t_2'\le t_2$ be such that $\{d\in D:\mu(d)\neq\varnothing\}\subseteq D^{t_1'},\{e\in E:\mu(e)\neq\varnothing\}\subseteq E^{t_2'}$. Then $\mu\in T_{<q_E}(\bar{M}^{(t_1',t_2')})$.

Lemma 3.5 Let $(t_1,t_2)\in \boldsymbol{d}\times\boldsymbol{e}$, and $(t_1',t_2')\in \boldsymbol{d}\times\boldsymbol{e}$ be such that $t_1'\le t_1$ y $t_2'\le t_2$. Then either $T_{<q_E}(\bar{M}^{(t_1,t_2)})\subseteq T_{<q_E}(\bar{M}^{(t_1',t_2')})=\varnothing$ or $T_{q_E}(\bar{M}^{(t_1,t_2)})\cap T_{q_E}(\bar{M}^{(t_1',t_2')})=\varnothing$.

Lemma 3.6 Let $(t_1,t_2)\in \boldsymbol{d}\times\boldsymbol{e}$, and $(t_1',t_2')\in \boldsymbol{d}\times\boldsymbol{e}$ be such that $T_{q_E}(\bar{M}^{(t_1,t_2)})\cap T_{q_E}(\bar{M}^{(t_1',t_2')})=\varnothing$. Then exists $(\bar{t}_1,\bar{t}_2)\in \boldsymbol{d}\times\boldsymbol{e}$ such that $T_{<q_E}(\bar{M}^{(t_1,t_2)})\subseteq T_{<q_E}(\bar{M}^{(\bar{t}_1,\bar{t}_2)})$ and $T_{<q_E}(\bar{M}^{(t_1',t_2')})\subseteq T_{<q_E}(\bar{M}^{(\bar{t}_1,\bar{t}_2)})$.

The proof of the previous lemmas is presented in the Appendix.

Proposition 3.4 Let $\bar{M}_U^{q_E}=(\bar{M},R_U,q_E)$ be a matching market with quota. Then $T_{q_E}(\bar{M})\subseteq S_G(\bar{M}_U^{q_E})$.

The proof the of this proposition is in the Appendix.

Now, we are going to show that the set of q_E-stable-G matching is non-empty.

Theorem 3.1 Let $\bar{M}_U^{q_E}=(\bar{M},R_U,q_E)$ be a matching market with quota. Then $S_G(\bar{M}_U^{q_E})\neq\varnothing$.

Proof. Let μ be a stable matching on $\bar{M}_U^{q_E}$

If $\#_E\mu=q_E$; clearly the matching $\mu\in S_G(\bar{M}_U^{q_E})$.

If $\#_E\mu\neq q_E$, we are going to consider the following cases:

Case 1: $\#_E\mu<q_E$

Let $(t_1,t_2)\in \boldsymbol{d}\times\boldsymbol{e}$, be the minimum such that $\mu(E)\subset D^{t_1}y\mu(D)\subset E^{t_2}$. Because

$\mu \in S(\bar{M}^{(t_1,t_2)})$ and every pairs *(d,e)* such that *μ(d)* = *μ(e)* = ∅, are not mutually acceptable and $e \notin Ch(\mu(d) \cup \{e\}, P_d)$, otherwise *(d,e)* block a *μ*, so, then $\mu \in T_{<q_E}(\bar{M}^{(t_1,t_2)})$. What and $T_{<q_E}(\bar{M}^{(t_1,t_2)}) \subseteq T_{<q_E}(\bar{M})$, by proposition $T_{<q_E}(\bar{M}^{(t_1,t_2)}) \subseteq S_G(\bar{M}_U^{q_E})$. This implies $\mu \in S_G(\bar{M}_U^{q_E})$.

Case 2: $\#_E \mu > q_E$.

Let $(t_1, t_2) \in \boldsymbol{d} \times \boldsymbol{e}$, such that $\mu \in S(\bar{M}^{(t_1,t_2)})$. By Theorem 2.3, for every $\mu' \in S(\bar{M}^{(t_1,t_2)})$, we have that $\#_E \mu' = \#_E \mu$. Consider the following sequence of matching $\mu^1, \ldots \mu^t, \mu^{t+1}, \ldots \mu^k$ such that;

- $\mu^1 \in S(\bar{M}^{(1,1)})$ and
- If $\mu^t \in S(\bar{M}^{(s_1,s_2)})$; then either $\mu^{t+1} \in S(\bar{M}^{(s_1+1,s_2)})$ or $\mu^{t+1} \in S(\bar{M}^{(s_1+1,s_2)})$.

By Lemma 2.1, we have that:

$$\#_E \mu^{t-1} \leq \#_E \mu^t \leq \#_E \mu^{t-1} + 1$$

$$\text{and} \quad \#_E \mu^1 \leq \ldots \leq \#_E \mu^{t-1} \leq \#_E \mu^t \leq \#_E \mu^{t-1} + 1 \leq \ldots \leq \#_E \mu$$

This implies that either $\#_E \mu^{t-1} = \#_E \mu^t$ or $\#_E \mu^{t-1} = \#_E \mu^t - 1$, for every *t*. Because $\#\mu > q_E$ and $\#_E \mu^1 \leq 1$, we have that there exists $\hat{t}$ such that $\#_E \mu^{\hat{t}} = q_E$ and $\mu^{\hat{t}}$ is stable on the market $\bar{M}^{(t_1,t_2)}$, this is $\mu^{\hat{t}} \in S(\bar{M}^{(t_1,t_2)})$ and $\#_E \mu^{\hat{t}} \in q_E$.

This implies: $\mu^{\hat{t}} \in T_{q_E}(\bar{M}^{(t_1,t_2)})$. By Proposition 3.1 we have that $T_{q_E}(\bar{M}^{(t_1,t_2)}) \subseteq T_{q_E}(\bar{M})$ and by $T_{q_E}(\bar{M}) \subseteq S_G(\bar{M}_U^{q_E})$ and by previous statement $\mu^{\hat{t}} \in S_G(\bar{M}_U^{q_E})$. Therefore $S_G(\bar{M}_U^{q_E}) \neq \emptyset$.

3.3 Characterization of the set of matchings q_E-stables-G

The following theorem is a complete characterization of the $\boldsymbol{q_e}$-stables-G sets $S_G(\bar{M}_U^{q_E})$.

Theorem 3.2 Let $\bar{M}_U^{q_E} = (\bar{M}, R_U, q_E)$ be a matching market with quota. Then $S_G(\bar{M}_U^{q_E}) = T_{<q_E}(\bar{M}) \cup T_{q_E}(\bar{M})$.

Proof. By Proposition 3.2 $T_{q_E}(\bar{M}) \subseteq S_G(\bar{M}_U^{q_E})$, by Proposition 3.4 $T_{q_E}(\bar{M}) \subseteq S_G(\bar{M}_U^{q_E})$ then it follows that: $T_{<q_E}(\bar{M}) \cup T_{q_E}(\bar{M}) \subseteq S_G(\bar{M}_U^{q_E})$.

We now demonstrate that $S_G(\bar{M}_U^{q_E}) \subseteq T_{<q_E}(\bar{M}) \cup T_{q_E}(\bar{M})$.

Be $\mu \in S_G(\bar{M}_U^{q_E}), (t_1, t_2) \in \boldsymbol{d} \times \boldsymbol{e}$ minimal such that $\mu(E) \subset D^{t_1}$ and $\mu(D) \subset E^{t_2}$.

Let us show that $\mu \in T_{q_E}(\bar{M})$ or $\mu \in T_{<q_E}(\bar{M})$.

As $\mu \in S_G(\bar{M}_U^{q_E}), \#_E \mu \leq q_E$, then we consider the cases in which $\#_E \mu < q_E$ and $\#_E \mu = q_E$.

If $\#_E \mu < q_E$, we prove that $\mu \in T_{<q_E}(\bar{M}^{(t_1,t_2)})$. Suppose that $\mu \notin T_{<q_E}(\bar{M}^{(t_1,t_2)})$. By definition $\mu \in T_{<q_E}(\bar{M}^{(t_1,t_2)})$ we have the following cases:

Case 1: $\mu \notin S(\bar{M}^{(t_1,t_2)})$

In this case there exists a blocking pair $(d,e) \in D^{t_1} \times E^{t_2}$, which is pair q_E-blocking-G for *μ* over $\bar{M}_U^{q_E}$ and this contradicts that $\mu \in S_G(\bar{M}_U^{q_E})$.

Case 2: $\mu \in S(\bar{M}^{(t_1,t_2)})$ and exists $d \notin \mu(E^{t_2})$ and $e \notin \mu(D^{t_1})$ are mutually acceptable.

As $d \notin \mu(E^{t_2})$ and $e \notin \mu(D^{t_1})$ this follows that *μ(d)* = ∅ = *μ(e)*. Since $\#_E \mu < q_E$, exists *E'={e}* and the matching $\mu_{(d,e)}^{E'}$ is the matching containing the assigned pairs in the matching *μ*, to which the pair *(d, e)* is added; therefore, $B_\mu \subset B_{\mu_{(d,e)}^{E'}}$ and by the condition v) responsive extension $\mu_{(d,e)}^{E'} R_U \mu$.

Moreover, since *d* and *e* are mutually acceptable, and taking into account the definition of the matching $\mu_{(d,e)}^{E'}$, it is verified $\mu_{(d,e)}^{E'}$ is q_E-individually rational. For previous statements, the pair *(d,e)* q_E-blocking-G to *μ* and this contradicts that $\mu \in S_G(\bar{M}_U^{q_E})$.

Case 3: $\mu \in S(\bar{M}^{(t_1,t_2)})$, for all $d \notin \mu(E^{t_2})$ and $e \notin \mu(D^{t_1})$ are not mutually acceptable and exists $(d,e) \in \mu(E^{t_2}) \times E \setminus \mu(D^{t_1})$ such that $e \in Ch(\mu(d) \cup \{e\}, P_d)$.

As $(d,e) \in \mu(E^{t_2}) \times E \setminus \mu(D^{t_1})$, such that $e \in Ch(\mu(d) \cup \{e\}, P_d)$, exists $E' = \mu(d) \cup \{e\}$ and the matching $\mu^{E'}_{(d,e)}$ is such that $B_\mu \subset B_{\mu^{E'}_{(d,e)}}$ then by the condition v) of responsive extension $\mu^{E'}_{(d,e)} R_U \mu$.

Moreover, as $e \in Ch(\mu(d) \cup \{e\}, P_d)$, and taking into account the definition of the matching, $\mu^{E'}_{(d,e)}$ it is verified: $\mu^{E'}_{(d,e)}$ is q_E-individually rational. For previous statements, the pair *(d,e)* q_E-blocking-G to μ and this contradicts that $\mu \in S_G(\bar{M}^{q_E}_U)$.

Finally, if $\#_E\mu = q_E$ as $\mu \in S_G(\bar{M}^{q_E}_U)$ then μ is not q_E-blocking-G by any pair of agents, therefore μ is a stable matching at some $\bar{M}^{(t_1,t_2)}$. That is, $\mu \in S(\bar{M}^{(t_1,t_2)})$ and $\#_E\mu = q_E$, which implies that $\mu \in T_{q_E}(\bar{M}^{(t_1,t_2)}) \subseteq T_{q_E}(\bar{M})$.

We can the conclude that:

$$S_G(\bar{M}^{q_E}_U) \subseteq T_{<q_E}(\bar{M}) \cup T_{q_E}(\bar{M})$$

Of the two inclusions shown we can say:

$$S_G(\bar{M}^{q_E}_U) = T_{<q_E}(\bar{M}) \cup T_{q_E}(\bar{M})$$

4. Comments and Conclusions

Among the different examples of many-to-one markets and matching problems linked to them are those of institutions subsidized by the state and the employees to be hired. The characteristics of this market generate problems that affect mainly those groups of competent low-income workers. Because of this, it is necessary to design long term integral strategies to produce equitable solutions for both the institutions and the workers; for this purpose, state actions should focus exclusively on sections qualified for certain tasks, which currently do not have access to work in institutions. Now, the state budget is limited and, as a consequence, it is often not possible to carry out all the possible matchings between institutions and workers that ask for that benefit. In other words, this model consists of a set of institutions, a set of workers and the state. Each institution has preferences for potential workers, each potential worker has preferences for potential companies, and the state has a priority over the possible "company-workers" pairs that can be agreed on.

This new model solves the problem in which the companies and the workers match with each other in such a way that they satisfy a stability property that depends on the preferences expressed by the participants and the state's preference. This property consists of no worker (company) having to work (hire) for an institution (workers) he cannot, or he does not want to work for. In addition, there is no "company-workers" pair preferring to reach an agreement different from the one assigned by the state; finally, all the "company-workers" pairs which reach an agreement are accepted by the state and do not exceed the budget it has. When this does not happen, the "company-workers" pair is said to block the matching. Besides, a solution is presented to problems such as the state's budget cuts or the assignment of money to other services for different reasons - global financial crisis, Covid 19 pandemic, etc. In this context, the assignments granted have to be interrupted and the new ones have to satisfy the stability property.

This work guarantees that the state's actions to give solutions in matters of work in accordance with workers' qualifications, with a limited budget, can be carried out with success for both the state and those who have access to the benefit. In other words, it is feasible to find solutions immune to the possibility of companies and workers not agreeing on the benefit distribution, or of the state not making a good distribution of the budget assigned. Even if the state's budget has to be cut, solutions as well as means to achieve them can be found. The difference from the jobs listed is that I now work in a many-to-one matching with responsive preferences for one side of agents. The previous ones are from the one-to-one matching and with responsive preferences for the two sets of agents. There are also many-to-one results with other types of agent preferences. Although the q_E-stability-G in this paper is guaranteed, the q_E-stability-R of the model has already been studied. Also, the many-to-many matching models with quota restriction are being studied.

Bibliography

1. Alkan, Ahmet(1986). Nonexistence of stable threesome Matchings. Mathematical Social Science, 16, 207-9.

2. D. Cantala (2004). "Restabilizing matching markets at senior level. Games and Economic Behavior, 48, 1-17.

3. Femenia, Delfina, Mabel Marí, Alejandro Neme y Jorge Oviedo. 2011. Stable solutions on matchings models with quota restrictions. International Game Theory Review, 13, 159-179.

4. Femenia, Delfina y Marí, Mabel (2012). "Application of the Game Theory to the Real-Estate Market with State Assistance. Journal of Mathematics and System Science, 2, 352-358.

5. Femenia, Delfina y Marí, Mabel (2020). "Stable solutions on many-to-one matching models with quota restriction with substitutable preferences for one side of agents". R. Bras. Eco. de Emp. 2020; 20(2): 79-93.

6. Gale, David y Lloyd Shapley. 1962. College admissions and the stability of marriage, American Mathematical Monthly, 69, 9-15.

7. Gale, David y Matilda Sotomayor. 1985. Some remarks on the stable matching problem, American Mathematical Monthly, 11, 223-32.

8. Marí, Mabel (2012). Assignment models with capacity restriction, path to stability, Brazilian Journal of Business Economics. RBEE. Vol 12 n°1, 7-19.

9. D. Gale y L. Shapley (1962). "College admissions and the stability of marriage", American Mathematical Monthly, 69, 9-15.

10. D. Gale y M. Sotomayor (1985). "Some remarks on the stable matching problem", American Mathematical Monthly, 11, 223-32.

11. Roth, Alvin y Marilda Sotomayor (1990). Two-sided Matching: a study in game-theoretic modeling and analysis. Cambridge University Press, Cambridge, England. Econometrica Society Monographs 18.

Appendix

Lemma A.1 Let $\mu \in T_{q_E}(\overline{M}^{(t_1,t_2)})$ and $(t_1',t_2') \in \mathbf{d} \times \mathbf{e}$ be such that $t_1' \leq t_1$ and $t_2' \leq t_2$, $\mu(D) \subseteq D^{t_1'}$ and $\mu(E) \subseteq E^{t_2'}$. Then $\mu \in T_{q_E}(\overline{M}^{(t_1',t_2')})$.

Proof. Because $\mu(D) \subseteq D^{t_1'}$ *and* $\mu(E) \subseteq E^{t_2'}$, μ is a matching on $\overline{M}^{(t_1',t_2')}$.

Let's suppose $\mu \notin S(\overline{M}^{(t_1',t_2')})$, then there is an agent or a pair of agents that blocks μ on $\overline{M}^{(t_1',t_2')}$. $E^{t_2'} \subseteq E^{t_2}$ and $D^{t_1'} \subseteq D^{t_1}$, then the agent or the pair of agents that blocks at μ on $\overline{M}^{(t_1',t_2')}$ is an agent or a pair of agents that blocks μ on $\overline{M}^{(t_1,t_2)}$.This contradicts the hypothesis, therefore $\mu \in S(\overline{M}^{(t_1',t_2')})$ and we can conclude that $\mu \in T_{q_E}(\overline{M}^{(t_1',t_2')})$. ∎

Lemma A.2 Let $(t_1,t_2) \in \mathbf{d} \times \mathbf{e}$, and $(t_1',t_2') \in \mathbf{d} \times \mathbf{e}$ be such that $t_1' \leq t_1$ y $t_2' \leq t_2$. Then either

$$T_{q_E}(\overline{M}^{(t_1,t_2)}) \subseteq T_{q_E}(\overline{M}^{(t_1',t_2')}) \text{ or } T_{q_E}(\overline{M}^{(t_1,t_2)}) \cap T_{q_E}(\overline{M}^{(t_1',t_2')}) = \emptyset.$$

Proof. Assume that $T_{q_E}(\overline{M}^{(t_1,t_2)}) \cap T_{q_E}(\overline{M}^{(t_1',t_2')}) \neq \emptyset$; we are going to show that $T_{q_E}(\overline{M}^{(t_1,t_2)}) \subseteq T_{q_E}(\overline{M}^{(t_1',t_2')})$.

If $\mu \in T_{q_E}(\overline{M}^{(t_1,t_2)}) \cap T_{q_E}(\overline{M}^{(t_1',t_2')})$, then $\mu \in T_{q_E}(\overline{M}^{(t_1,t_2)})$ and $\mu \in T_{q_E}(\overline{M}^{(t_1',t_2')})$.

Since $\mu \in S(\overline{M}^{(t_1',t_2')})$, that is, the agents of D that are not singles, are in $D^{t_1'}$and the agents of E that are not singles, are in $E^{t_2'}$; formally:

$$\mu(D) \subseteq D^{t_1'} \text{ and } \mu(E) \subseteq E^{t_2'}. \quad \text{(A1)}$$

Let be amatching $\bar{\mu} \in T_{q_E}(\overline{M}^{(t_1,t_2)})$; we will show that $\bar{\mu} \in T_{q_E}(\overline{M}^{(t_1',t_2')})$.

Since $\bar{\mu} \in T_{q_E}(\overline{M}^{(t_1,t_2)})$ and furthermore, $\mu \in T_{q_E}(\overline{M}^{(t_1,t_2)})$, then μ and $\bar{\mu}$ are both stable in the $\overline{M}^{(t_1,t_2)}$ model. Therefore, since P_d is q_d-responsive, by Theorem 2.3, both matchings have the same set of singles, then we obtain that:

$$\{d \in D: \bar{\mu}(d) \neq \emptyset\} = \{d \in D: \mu(d) \neq \emptyset\} \subseteq D^{t_1'},$$

$$\{e \in E: \bar{\mu}(e) \neq \emptyset\} = \{e \in E: \mu(e) \neq \emptyset\} \subseteq E^{t_2'}.$$

By Lemma A.1, we can conclude that $\bar{\mu} \in T_{q_E}(\overline{M}^{(t_1',t_2')})$. ∎

Lemma A.3 Let $(t_1,t_2)\in\mathbf{d}\times\mathbf{e}$, $(t_1',t_2')\in\mathbf{d}\times\mathbf{e}$ be such $T_{q_E}\big(\overline{M}^{(t_1,t_2)}\big)\cap T_{q_E}\big(\overline{M}^{(t_1',t_2')}\big)\neq\emptyset$. Then exists $(\overline{t_1},\overline{t_2})\in \boldsymbol{d}\times\mathbf{e}$ such that

$$T_{q_E}\big(\overline{M}^{(t_1,t_2)}\big)\subseteq T_{q_E}\big(\overline{M}^{(\bar t_1,\bar t_2)}\big) \text{ and } \big(\overline{M}^{(t_1',t_2')}\big)\subseteq T_{q_E}\big(\overline{M}^{(\bar t_1,\bar t_2)}\big).$$

Proof. Let $\mu\in T_{q_E}\big(\overline{M}^{(t_1,t_2)}\big)\cap T_{q_E}\big(\overline{M}^{(t_1',t_2')}\big)$. Let $(\bar t_1,\bar t_2)\in\mathbf{d}\times\mathbf{e}$ be, such that $\bar t_1=\min\{t_1,t_1'\}$ and $\bar t_2=\min\{t_2,t_2'\}$.

Since $\mu\in T_{q_E}\big(\overline{M}^{(t_1,t_2)}\big)$, we have that

$$\{d\in D:\mu(d)\neq\emptyset\}\subseteq D^{t_1} \text{ and } \{e\in E:\mu(e)\neq\emptyset\}\subseteq E^{t_2}.$$

Because $\mu\in T_{q_E}\big(\overline{M}^{(t_1',t_2')}\big)$, we have that

$$\{d\in D:\mu(d)\neq\emptyset\}\subseteq D^{t_1'} \text{ and } \{e\in E:\mu(e)\neq\emptyset\}\subseteq E^{t_2'}.$$

Hence we have that

$$\{d\in D:\mu(d)\neq\emptyset\}\subseteq D^{\bar t_1} \text{ and } \{e\in E:\mu(e)\neq\emptyset\}\subseteq E^{\bar t_2}.$$

By Lemma A1 $\mu\in T_{q_E}\big(\overline{M}^{(\bar t_1,\bar t_2)}\big)$.

Which implies that

$$T_{q_E}\big(\overline{M}^{(t_1,t_2)}\big)\cap T_{q_E}\big(\overline{M}^{(\bar t_1,\bar t_2)}\big)\neq\emptyset \text{ and } T_{q_E}\big(\overline{M}^{(t_1',t_2')}\big)\cap T_{q_E}\big(\overline{M}^{(\bar t_1,\bar t_2)}\big)\neq\emptyset.$$

Then Lemma A.2 implies that

$$T_{q_E}\big(\overline{M}^{(t_1,t_2)}\big)\subseteq T_{q_E}\big(\overline{M}^{(\bar t_1,\bar t_2)}\big) \text{ and } T_{q_E}\big(\overline{M}^{(t_1',t_2')}\big)\subseteq T_{q_E}\big(\overline{M}^{(\bar t_1,\bar t_2)}\big).$$

The above lemmas allow us to demostrate:

Proof A.1 Given $M_U=(M,R_U)$, $(t^*,s^*)\in\mathbf{d}\times\mathbf{e}$, there exists $K\subseteq\mathbf{d}\times\mathbf{e}$, such that

$$T_q(\overline{M})=\dot{\bigcup_{(t^*,s^*)\in K}} T_q\big(\overline{M}^{(t^*,s^*)}\big)$$

Proof. Let $K=\{(t_1,t_2)\in\mathbf{d}\times\mathbf{e}:\forall\ (t_1',t_2')\neq(t_1,t_2)t_1'\leq t_1t_2'\leq t_2$ such that $T_{q_E}\big(\overline{M}^{(t_1,t_2)}\big)\cap T_{q_E}\big(\overline{M}^{(t_1',t_2')}\big)=\emptyset\}$.

First we show that $T_{q_E}(\overline{M})=\bigcup_{(t_1,t_2)\in K}T_{q_E}\big(\overline{M}^{(t_1,t_2)}\big)$ and second that such union is disjoint.

By definition of $T_q(M)$, we have that:

$$\bigcup_{(t_1,t_2)\in K} T_{q_E}\big(\overline{M}^{(t_1,t_2)}\big)\subseteq T_{q_E}(\overline{M}) \qquad (A2)$$

We are going to show that

$$T_{q_E}(\overline{M})\subseteq\bigcup_{(t_1,t_2)\in K} T_{q_E}\big(\overline{M}^{(t_1,t_2)}\big)$$

Let $\mu\in T_{q_E}(\overline{M})=\bigcup_{(t_1,t_2)\in N}T_{q_E}\big(\overline{M}^{(t_1,t_2)}\big)$ by, then exists $(t_1,t_2)\in\mathbf{d}\times\mathbf{e}$ such that $\mu\in T_{q_E}\big(\overline{M}^{(t_1,t_2)}\big)$.

Assume that $(t_1,t_2)\notin K$ then, exists $(t_1',t_2')\neq(t_1,t_2)$, such that $T_q\big(\overline{M}^{(t_1,t_2)}\big)\cap T_q\big(\overline{M}^{(t_1',t_2')}\big)\neq\emptyset$.

Let $(t_1^*, t_2^*) \in \mathbf{d}\times\mathbf{e}$ by, the minimal pair such that

$$\mu \in T_{q_E}(\bar{M}^{(t_1^*,t_2^*)}), \tag{A3}$$

That is, for every $(\bar{t}_1, \bar{t}_2) \neq (t_1^*, t_2^*), \bar{t}_1 \leq t_1^*$ y $\bar{t}_2 \leq t_2^*$ which implies that $T_{q_E}(\bar{M}^{(t_1^*,t_2^*)}) \cap T_{q_E}(\bar{M}^{(\bar{t}_1,\bar{t}_2)}) = \emptyset$.

We are going to show that $(t_1^*, t_2^*) \in K$. Assume otherwise, that $(t_1^*, t_2^*) \notin K$, then exists $(t_1', t_2') \neq (t_1^*, t_2^*)$, with $t_1' \leq t_1^*$, $t_2' \leq t_2^*$ and $T_{q_E}(\bar{M}^{(t_1',t_2')}) \cap T_{q_E}(\bar{M}^{(t_1^*,t_2^*)}) \neq \emptyset$.

Por Lema A.2

$$T_{q_E}(\bar{M}^{(t_1^*,t_2^*)}) \subseteq T_{q_E}(\bar{M}^{(t_1',t_2')}). \tag{A4}$$

By (A3) and (A4), $\mu \in T_{q_E}(\bar{M}^{(t_1',t_2')})$ but this contradict the minimality of (t_1^*, t_2^*).

Consequently $\mu \in T_{q_E}(\bar{M}^{(t_1^*,t_2^*)})$, with $(t_1^*, t_2^*) \in K$, thus

$$\mu \in \bigcup_{(t_1,t_2)\in K} T_{q_E}(\bar{M}^{(t_1,t_2)}).$$

Hence we have

$$T_{q_E}(\bar{M}) \subseteq \bigcup_{(t_1,t_2)\in K} T_{q_E}(\bar{M}^{(t_1,t_2)})$$

By (2),

$$T_{q_E}(\bar{M}) = \bigcup_{(t_1,t_2)\in K} T_{q_E}(\bar{M}^{(t_1,t_2)}).$$

Now, we are going to show that such union disjoint.

Assume otherwise, $(t_1', t_2') \in K$ and $(t_1'', t_2'') \in K$, such that $T_{q_E}(\bar{M}^{(t_1',t_2')}) \cap T_{q_E}(\bar{M}^{(t_1'',t_2'')}) \neq \emptyset$.

By Lemma A.3 exists $(\bar{t}_1, \bar{t}_2) \in \mathbf{d}\times\mathbf{e}$, with $\bar{t}_i \leq \min\{t_i', t_i''\}$, $i = 1,2$ such that

$$T_{q_E}(\bar{M}^{(t_1',t_2')}) \subset T_{q_E}(\bar{M}^{(\bar{t}_1,\bar{t}_2)}) \ \ and \ T_{q_E}(\bar{M}^{(t_1'',t_2'')}) \subset T_{q_E}(\bar{M}^{(\bar{t}_1,\bar{t}_2)}),$$

Contradicting $(t_1', t_2') \in K$ and $(t_1'', t_2'') \in K$, which concludes proof.

■

Proposition A.4 Let $\bar{M}_U^{q_E} = (\bar{M}, R_U, q_E)$ be a matching market with quota restriction. Then

$$T_{q_E}(\bar{M}) \subseteq S(\bar{M}_U^{q_E}).$$

Proof. Let $\mu \in T_{q_E}(\bar{M})$ by and assume that $\mu \notin S(\bar{M}_U^{q_E})$. Exists (t_1, t_2): $\mu \in T_{q_E}(\bar{M}^{(t_1,t_2)})$. Since $\mu \in S(\bar{M}^{(t_1,t_2)})$ and $\#_E\mu = q_E$, we have that μ is q_E-individually rational. Let (d, e) by a q_E-blocking pairs to μ, that is:

1. $e \notin \mu(d)$, $dP_e\mu(e)$, $e \in Ch(\mu(d) \cup \{e\}, P_d)$ and
2. either
   (a) $\#_E\mu(d) = q_d$ and $\mu(e) \in D$ or
   (b) exists $E' \subseteq \mu(d) \cup \{e\}$ such that $e \in E'$, $\mu_{(d,e)}^{E'}$ is q_E −individually rational and $\mu_{(d,e)}^{E'} R_U \mu$.

We are going to consider the following cases:

Case 1. $d \in D^{t_1}$ and $e \in E^{t_2}$

By condition 1. $\mu \notin S(\overline{M}^{(t_1,t_2)})$, a contradiction is rached.

Case 2. $d \notin D^{t_1}$ and $e \in E^{t_2}$

As $d \notin D^{t_1}$, then $d \notin \mu(E)$.

If $\mu(e) = \emptyset$, then $E' = \{e\}$ it is verified $\#_E \mu^{E'}_{(d,e)} > q_E$, since $\#_E \mu = q_E$ and contradicts condition 2. b). Then it is $\mu(e) \neq \emptyset$

Since $\mu(e) \neq \emptyset$, then $\mu(e) \in D^{t_1}$ and since $d \notin D^{t_1}$, it is verified

$$\mu(e) \succ_D d. \qquad \text{(A5)}$$

There exists only $E' = \{e\}$ for which $\mu^{E'}_{(d,e)}$ is q_E-individually rational, being:

$$\mu^{E'}_{(d,e)}(f) = \begin{cases} \mu(f) & if \quad f \notin \{d, e, \mu(e)\} \\ Ch(E', P_f) & if \quad f = d \\ d & if \quad f = e \\ Ch(\mu(f) \backslash \{e\}, P_f) & if \quad f = \mu(e) \\ \emptyset & if \quad otherwise \end{cases}$$

Then,

$$\mu(D) = \mu^{E'}_{(d,e)}(D) \quad and \quad \mu^{E'}_{(d,e)}(E) = \mu(E) \cup \{d\}. \qquad \text{(A6)}$$

Moreover, for all $d' \in D \backslash \{d, \mu(e)\}$ it is verified $\#\mu(\mu(e)) > \#\bar{\mu}(\mu(e))$ and $\#\bar{\mu}(d) > \#\mu(d)$. Then, from (A5), (A6) and condition $viii)$ of R_U responsive, $\mu P_U \mu^{E'}_{(d,e)}$, contradicts that (d,e) q_E-blocks μ.

Case 3. $d \in D^{t_1}$ and $e \notin E^{t_2}$

As $e \notin E^{t_2}$, then $e \notin \mu(D)$.

If $\mu(d) = \emptyset$ one arrives idem Case 2. Contradiction, then it is $\mu(d) \neq \emptyset$

Since $\mu(d) \neq \emptyset$, then for all $e' \in \mu(d)$, $E' \in E^{t_2}$ and then since $e \notin E^{t_2}$, it is verified

$$e' \succ_E e. \qquad \text{(A7)}$$

There exists only $E' = \mu(d) \backslash \{e' \cup \{e\}$ for which $\mu^{E'}_{(d,e)}$ is q_E-individually rational.

From (A7) and because $\succ_{2^E}$ is responsive, is verified $\mu(d) \succ_{2^E} E'$.

Considering:

$$\bar{\mu} = \mu^{E'}_{(d,e)}(f) = \begin{cases} \mu(f) & if \quad f \notin \{d, e\} \cup \mu(e)\} \\ E' & if \quad f = d \\ d & if \quad f = e \end{cases}$$

Since $\mu(E) = \bar{\mu}(E)$, $\mu(D) = \bar{\mu}(D) \backslash e' \cup e$ and $e' \succ_E e$ then, by item $vii)$ of R_U responsive verifies $\bar{\mu} P_U \mu$ and contradicts that (d,e) q_E-blocking μ.

Case 4. $d \notin D^{t_1}$ and $e \notin E^{t_2}$

Since $d \notin D^{t_1}$ and $e \notin E^{t_2}$, then $\mu(d) = \mu(e) = \emptyset$, then for all $E' : e \in E'$, $\#_E \mu^{E'}_{(d,e)} < q_E$ and is not q_E-individually rational.

■

Lemma A.5 Let $\mu \in T_{<q_E}(\overline{M}^{(t_1,t_2)})$, ($t_1$, t_2) ∈**e** x **d** and $t_1' \leq t_1$ y $t_2' \leq t_2$ be such that $\{d \in D: \mu(d) \neq \emptyset\} \subseteq D^{t_1'}$, $\{e \in E: \mu(e) \neq \emptyset\} \subseteq E^{t_2'}$. Then $\mu \in T_{<q_E}(\overline{M}^{(t_1',t_2')})$

Proof. Let $\mu \in T_{<q_E}(\overline{M}^{(t_1,t_2)})$; since $\{d \in D: \mu(d) \neq \emptyset\} \subseteq D^{t_1'}$ and $\{e \in E: \mu(e) \neq \emptyset\} \subseteq E^{t_2'}$, then μ is a matching in $\overline{M}^{(t_1',t_2')}$.

Moreover $E^{t_2'} \subseteq E^{t_2}$ and $D^{t_1'} \subseteq D^{t_1}$, then any agent or pair of agents that blocks μ in $\overline{M}^{(t_1',t_2')}$ is an agent or pair of agents that blocks μ at $\overline{M}^{(t_1,t_2)}$. Then $\mu \in S(\overline{M}^{(t_1',t_2')})$.

By hypothesis $\mu \in T_{<q_E}(\overline{M}^{(t_1,t_2)})$, then $\#\mu < q$ and for all $(d,e) \in D \backslash \mu(E^{t_2}) \times E \backslash \mu(D^{t_1})$, d and e are not mutually acceptable.

Since $\mu(E^{t_2}) = \mu(E^{t_2'})$ and $\mu(D^{t_1}) = \mu(D^{t_1'})$, then for all $(d,e) \in D \backslash \mu(E^{t_2'}) \times E \backslash \mu(D^{t_1'})$, d and e are not mutually acceptable and $e \notin Ch(\mu(d) \cup e, P_d)$ for all $(d,e) \in \mu(E^{t_2}) \times E \backslash \mu(D^{t_1})$.

Therefore $\mu \in T_{<q_E}(\overline{M}^{((t_1',t_2')})$.

■

Lemma A.6 Let $(t_1,t_2) \in \mathbf{d} \times \mathbf{e}$, and $(t_1',t_2') \in \mathbf{d} \times \mathbf{e}$ be such that $t_1' \leq t_1$ y $t_2' \leq t_2$. Then either

$$T_{<q_E}\left(\overline{M}^{(t_1,t_2)}\right) \subseteq T_{<q_E}\left(\overline{M}^{((t_1',t_2')}\right) \text{ or } T_{<q_E}(\overline{M}^{(t_1,t_2)}) \cap T_{<q_E}(\overline{M}^{(t_1',t_2')}) = \emptyset.$$

Proof. Be $(t_1,t_2) \in \mathbf{d} \times \mathbf{e}$, and $(t_1',t_2') \in \mathbf{d} \times \mathbf{e}$, such that $t_1' \leq t_1$ *and* $t_2' \leq t_2$.

Suppose that $T_{<q_E}(\overline{M}^{(t_1,t_2)}) \cap T_{<q_E}(\overline{M}^{(t_1',t_2')}) \neq \emptyset$; let us show that

$$T_{<q_E}(\overline{M}^{(t_1,t_2)}) \subseteq T_{<q_E}(\overline{M}^{(t_1',t_2')}).$$

If $\mu \in T_{<q_E}(\overline{M}^{(t_1,t_2)}) \cap T_{<q_E}(\overline{M}^{(t_1',t_2')})$. Since $\mu \in T_{<q_E}(\overline{M}^{(t_1',t_2')})$, i.e., all $d \in D$ that are not singles are in $D^{t_1'}$ and $e \in E$ are not singles are in $E^{t_2'}$; formally.

$$\{d \in D: \mu(d) \neq \emptyset\} \subseteq D^{t_1'} \text{ and } \{e \in E: \mu(e) \neq \emptyset\} \subseteq E^{t_2'}. \quad \text{(A8)}$$

Let $\bar{\mu} \in T_{<q_E}(\overline{M}^{(t_1,t_2)})$ be a matching, let us show that $\bar{\mu} \in T_{<q_E}(\overline{M}^{(t_1',t_2')})$.

Since $\bar{\mu} \in T_{<q_E}(\overline{M}^{(t_1,t_2)})$ and furthermore, $\mu \in T_{<q_E}(\overline{M}^{(t_1',t_2')})$, we have that $\mu \in T_{<q_E}(\overline{M}^{(t_1,t_2)})$, then μ and $\bar{\mu}$ are both stable in the model $\overline{M}^{(t_1,t_2)}$. Therefore, by Theorem 1.5, both matchings have the same set of singles, and taking into account (A.8),

$$\{d \in D: \bar{\mu}(d) \neq \emptyset\} = \{d \in D: \mu(d) \neq \emptyset\} \subseteq D^{t_1'},$$

$$\{e \in E: \bar{\mu}(e) \neq \emptyset\} = \{e \in E: \mu(e) \neq \emptyset\} \subseteq E^{t_2'}.$$

Then, by (8) and Lemma A.5, we can deduce that$\bar{\mu} \in T_{<q_E}(\overline{M}^{(t_1',t_2')})$. ■

Lemma A.7 Let $(t_1,t_2) \in \boldsymbol{d} \times \boldsymbol{e}$, and $(t_1',t_2') \in \boldsymbol{d} \times \boldsymbol{e}$, such that $T_{<q_E}(\overline{M}^{(t_1,t_2)}) \cap T_{<q_E}(\overline{M}^{(t_1',t_2')}) \neq \emptyset$, then there exists $\left(\overline{t_1},\overline{t_2}\right) \in \mathbf{d} \times \mathbf{e}$ such that:

$$T_{<q_E}\left(\overline{M}^{(t_1,t_2)}\right) \subseteq T_{<q_E}\left(\overline{M}^{(\bar{t}_1,\bar{t}_2)}\right) \text{ and } T_{<q_E}(\overline{M}^{(t_1',t_2')}) \subseteq T_{<q_E}(\overline{M}^{(\bar{t}_1,\bar{t}_2)}).$$

Proof. Be $(t_1,t_2) \in \mathbf{d} \times \mathbf{e}$, $(t_1',t_2') \in \boldsymbol{d} \times \mathbf{e}$ and $\mu \in T_{<q_E}(\overline{M}^{(t_1,t_2)}) \cap T_{<q_E}(\overline{M}^{(t_1',t_2')})$.

Since $\mu \in T_{<q_E}(\overline{M}^{(t_1,t_2)})$, the agents of *D* that are not singles, are in D^{t_1}, and the agents of *E* that are not singles are in E^{t_2}, formally

$$\{d \in D: \mu(d) \neq \emptyset\} \subseteq D^{t_1} \text{ and } \{e \in E: \mu(e) \neq \emptyset\} \subseteq E^{t_2}. \quad \text{(A9)}$$

Analogously, since $\mu \in T_{<q_E}(\overline{M}^{(t_1',t_2')})$, the agents $d \in D$ that are not singles are in $D^{t_1'}$and those of E that are not singles are in $E^{t_2'}$; formally:

$$\{d \in D: \mu(d) \neq \emptyset\} \subseteq D^{t_1'} \text{ and } \{e \in E: \mu(e) \neq \emptyset\} \subseteq E^{t_2'}. \quad \text{(A10)}$$

We define $(\bar{t}_1,\bar{t}_2) \in \mathbf{d} \times \mathbf{e}$, such that $\bar{t}_1 = \min\{t_1,t_1'\}$ *and* $\bar{t}_2 = \min\{t_2,t_2'\}$.

Then, what is obtained in (A9) and (A10) we can express as follows:

$$\{d \in D: \mu(d) \neq \emptyset\} \subseteq D^{\bar{t}_1} \text{ and } \{e \in E: \mu(e) \neq \emptyset\} \subseteq E^{\bar{t}_2}. \quad \text{(A11)}$$

By the way we define $\bar{t}_1$ and $\bar{t}_2$, it is verified that$\bar{t}_1 \leq t_1$and $\bar{t}_2 \leq t_2$ and taking into account (A11), it is possible to apply Lemma 3.4, then

$$\mu \in T_{<q_E}(\bar{M}^{(\bar{t}_1,\bar{t}_2)}).$$

Resulting,

$$T_{<q_E}\left(\bar{M}^{(t_1,t_2)}\right) \cap T_{<q_E}\left(\bar{M}^{(\bar{t}_1,\bar{t}_2)}\right) \neq \emptyset \text{ and } T_q(\bar{M}^{(t_1',t_2')}) \cap T_{<q_E}(\bar{M}^{(\bar{t}_1,\bar{t}_2)}) \neq \emptyset.$$

Then, by Lemma A.6, $T_{<q_E}(\bar{M}^{(t_1,t_2)}) \subseteq T_{<q_E}(\bar{M}^{(\bar{t}_1,\bar{t}_2)})$ and $T_{<q_E}(\bar{M}^{(t_1',t_2')}) \subseteq T_{<q_E}(\bar{M}^{(\bar{t}_1,\bar{t}_2)})$.

Proposition A.2 *If* $\bar{M}_U^{q_E} = (\bar{M}, R_U, q_E)$ is a many-to-one model with capacity constraint and $\hat{K} = \{(t_1,t_2) \in \mathbf{e} \times \mathbf{d}: \forall\ (t_1',t_2') \neq (t_1,t_2) \quad t_1' \leq t_1\ t_2' \leq t_2 \quad such\ that \quad T_{<q_E}(\bar{M}^{(t_1,t_2)}) \cap T_{<q_E}(\bar{M}^{(t_1',t_2')}) = \emptyset\}$, then

$$T_{<q_E}(\bar{M}) = \dot{\bigcup_{(t_1,t_2)\in\hat{K}}} T_{<q_E}(\bar{M}^{(t_1,t_2)})$$

Proof. Let $\hat{K} = \{(t_1,t_2) \in \mathbf{e} \times \mathbf{d}$: for all $(t_1',t_2') \neq (t_1,t_2)$

$$t_1' \leq t_1, t_2' \leq t_2 \quad and \quad T_{<q_E}(\bar{M}^{(t_1,t_2)}) \cap T_{<q_E}(\bar{M}^{(t_1',t_2')}) = \emptyset\}.$$

We will first prove that

$$T_{<q_E}(M) = \dot{\bigcup_{(t_1,t_2)\in\hat{K}}} T_{<q_E}(\bar{M}^{(t_1,t_2)})$$

and then that such a union is disjunctive.

Since $\hat{K} \subset N$ by definition of the set $T_{<q_E}(M)$, it is verified:

$$\bigcup_{(t_1,t_2)\in\hat{K}} T_{<q_E}\left(\bar{M}^{(t_1,t_2)}\right) \subseteq T_{<q_E}(M) \quad (A12)$$

Let us now show that

$$T_{<q_E}(M) \subseteq \bigcup_{(t_1,t_2)\in\hat{K}} T_{<q_E}(\bar{M}^{(t_1,t_2)})$$

Let

$$\mu \in T_{<q_E}(M) = \bigcup_{(t_1,t_2)\in\hat{K}} T_{<q_E}(\bar{M}^{(t_1,t_2)})$$

Then ther exists $(t_1,t_2) \in \mathbf{e} \times \mathbf{d}$ such that $\mu \in T_{<q_E}(\bar{M}^{(t_1,t_2)})$.

If $(t_1,t_2) \in \hat{K}$, the equality is proved. If $(t_1,t_2) \notin \hat{K}$ then there exists $(t_1',t_2') \neq (t_1,t_2)$, such that $T_{<q_E}(\bar{M}^{(t_1,t_2)}) \cap T_{<q_E}(\bar{M}^{(t_1',t_2')}) \neq \emptyset$.

Let $(t_1^*,t_2^*) \in (\boldsymbol{d} \times \boldsymbol{e})$ be a minimal such that.

$$\mu \in T_{<q_E}(\bar{M}^{(t_1^*,t_2^*)}), \quad \text{(A13)}$$

i.e., for all $(\bar{t}_1,\bar{t}_2) \neq (t_1^*,t_2^*), \bar{t}_1 \leq t_1^*$ and $\bar{t}_2 \leq t_2^*$ it is verified that $T_{<q_E}(\bar{M}^{(t_1^*,t_2^*)}) \cap T_{<q_E}(\bar{M}^{(\bar{t}_1,\bar{t}_2)}) = \emptyset$.

Let us note that (t_1^*,t_2^*) always exists since $\mu \in T_{<q_E}(\bar{M}^{(t_1,t_2)})$. Moreover, if $(\bar{t}_1,\bar{t}_2)$ does not exist under the above conditions it is $(t_1,t_2) = (t_1^*,t_2^*)$ and otherwise, $(\bar{t}_1,\bar{t}_2) = (t_1^*,t_2^*)$.

Let's prove that $(t_1^*,t_2^*) \in \hat{K}$

Suppose that $(t_1^*,t_2^*) \not\subset \hat{K}$, then there exists $(t_1',t_2') \neq (t_1^*,t_2^*)$, being $t_1' \leq t_1^*$, $t_2' \leq t_2^*$ and $T_{<q_E}(\bar{M}^{(t_1',t_2')}) \cap T_{<q_E}(\bar{M}^{(t_1^*,t_2^*)}) \neq \emptyset$.

By Lemma A.5

$$T_{<q_E}(\overline{M}^{(t_1^*,t_2^*)}) \subseteq T_{<q_E}(\overline{M}^{(t_1',t_2')}). \quad \text{(A14)}$$

From (A13) and (A14), $\mu \in T_{<q_E}(\overline{M}^{(t_1',t_2')})$ and this contradicts the minimality of (t_1^*, t_2^*).

Therefore

$$\mu \in T_{<q_E}\left(\overline{M}^{(t_1^*,t_2^*)}\right) \subseteq \bigcup_{(t_1,t_2)\in\widehat{K}} T_{<q_E}(\overline{M}^{(t_1,t_2)})$$

With $(t_1^*, t_2^*) \in \widehat{K}$, then

$$T_{<q_E}(M) \subseteq \bigcup_{(t_1,t_2)\in\widehat{K}} T_{<q_E}\left(\overline{M}^{(t_1,t_2)}\right)$$

From (A12) and (A14) it is shown:

$$T_{<q_E}(M) = \bigcup_{(t_1,t_2)\in\widehat{K}} T_{<q_E}\left(\overline{M}^{(t_1,t_2)}\right)$$

It remains only to show that this union is disjoint.

Be$(t_1', t_2') \in \widehat{K}$ y $(t_1'', t_2'') \in \widehat{K}$.

Suppose that$T_{<q_E}(\overline{M}^{(t_1',t_2')}) \cap T_{<q_E}(\overline{M}^{(t_1'',t_2'')}) \neq \emptyset$.

By Lemma A.6 exists $(\bar{t}_1, \bar{t}_2) \in \mathbf{e} \times \mathbf{d}$, with $\bar{t}_i \leq \min\{t_i', t_i''\}$ for $i = 1{,}2$ such that

$$T_{<q_E}\left(\overline{M}^{(t_1',t_2')}\right) \subset T_{<q_E}\left(\overline{M}^{(\bar{t}_1,\bar{t}_2)}\right) \text{ and } T_{<q_E}(\overline{M}^{(t_1'',t_2'')}) \subset T_{<q_E}(\overline{M}^{(\bar{t}_1,\bar{t}_2)}),$$

contradicting this that$(t_1', t_2') \in K$ y $(t_1'', t_2'') \in \widehat{K}$.

Finally we conclude:

$$T_{<q_E}(\overline{M}) = \dot{\bigcup_{(t_1,t_2)\in\widehat{K}}} T_{<q_E}\left(\overline{M}^{(t_1,t_2)}\right)$$

■

Proposition A.8 Let $\overline{M}_U^{q_E} = (\overline{M}, R_U, q_E)$ be a matching market with quota. Then

$$T_{<q_E}(\overline{M}) \subseteq S(\overline{M}_U^{q_E}).$$

Proof. Let $\mu \in T_{<q_E}(\overline{M})$and suppose that $\mu \notin S(\overline{M}_U^{q_E})$.

Since $\mu \in T_{<q_E}(\overline{M})$, there exists (t_1, t_2): $\mu \in T_{<q_E}(\overline{M}^{(t_1,t_2)})$. Then $\mu \in S(\overline{M}^{(t_1,t_2)})$, $\#_E\mu < q_E$, d and e are not mutually acceptable for all $(d, e) \in D\backslash\mu(E^{t_2}) \times E\backslash\mu(D^{t_1})$ *and* $e \notin Ch(\mu(d) \cup e, P_d)$ for all $(d, e) \in \mu(E^{t_2}) \times E\backslash\mu(D^{t_1})\}$. Therefore μ is q_E-individually rational.

Ther exists(d, e) that q_E-blocks μ, then:

1. $e \notin \mu(d)$, $dP_e\mu(e)$, $e \in Ch(\mu(d) \cup \{e\}, P_d)$ and
2. is verified:

   (a) $\#\mu(d) = q_d$ and $\mu(e) \in D$ or

   (b) there exists $E' \subseteq \mu(d) \cup \{e\}$ such that $e \in E'$, $\mu_{(d,e)}^{E'}$ es q_E −individually rational and $\mu_{(d,e)}^{E'} R_U \mu$.

Since$(d, e) \in D \times E$, $D^{t_1} \subseteq D$ and $E^{t_2} \subseteq E$, let us consider the following cases:

Case 1. $d \in D^{t_1}$ and $e \in E^{t_2}$

From condition 1. it is verified $\mu \notin S(\overline{M}^{(t_1,t_2)})$, a contradiction is reached.

Case 2. $d \notin D^{t_1}$and $e \in E^{t_2}$

Since $d \notin D^{t_1}$, then $d \notin \mu(E)$

If $\mu(e) = \emptyset$, then $e \notin \mu(D^{t_1})$ and since $d \notin D^{t_1}$, then $d \notin \mu(E^{t_2})$, since d and e are not mutually acceptable, this contradicts condition 1. of. q_E-blocking. Then it is $\mu(e) \neq \emptyset$

Since $\mu(e) \neq \emptyset$, then $\mu(e) \in D^{t_1}$ and since $d \notin D^{t_1}$, is verified

$$\mu(e) \succ_D d. \tag{A15}$$

There exists only $E' = \{e\}$ for which $\mu_{(d,e)}^{E'}$ is q_E-individually rational:

$$\mu_{(d,e)}^{E'}(f) = \begin{cases} \mu(f) & if \quad f \notin \{d, e, \mu(e)\} \\ Ch(E', P_f) & if \quad f = d \\ d & if \quad f = e \\ Ch(\mu(f)\backslash\{e\}, P_f) & if \quad f = \mu(e) \\ \emptyset & if \quad otherwise \end{cases}$$

Then,

$$\mu(D) = \mu_{(d,e)}^{E'}(D) \ \ and \ \ \mu_{(d,e)}^{E'}(E) = \mu(E) \cup \{d\}. \tag{A16}$$

Moreover, for all $d' \in D\backslash\{d, \mu(e)\}$ $\#\mu(\mu(e)) > \#\bar{\mu}(\mu(e))$ and $\#\bar{\mu}(d) > \#\mu(d)$ are verified. Then, from (A15), (A16) and condition viii) of R_U responsive, $\mu P_U \mu_{(d,e)}^{E'}$, contradicts that (d, e) q_E-blocks μ.

Case 3. $d \in D^{t_1}$ and $e \notin E^{t_2}$

As $e \notin E^{t_2}$, then $e \notin \mu(D)$

If $\mu(d) = \emptyset$ one arrives idem Case 2 (because *d* and *e* are not mutually acceptable). Contradiction, then it is $\mu(d) \neq \emptyset$.

Since $\mu(d) \neq \emptyset$, then $d \in \mu(E^{t_2})$.

Since $e \notin E^{t_2}$ then $e \in E\backslash\mu(D^{t_1})$. Then, $e \notin Ch(\mu(d) \cup e, P_d)$. It contradicts condition 1

Case 4. $d \notin D^{t_1}$ and $e \notin E^{t_2}$

Since $d \notin D^{t_1}$ and $e \notin E^{t_2}$, d and e are not mutually acceptable and contradicts condition 1.