

Nonlinear-differential evolution equation arising in option pricing when including transaction costs: a viscosity solution approach¹

Since the Black-Scholes paper (1973) presented a formula for pricing options, there has been an increasing interest in problems arising in Financial Mathematics and in particular in derivatives pricing. The standard approach to this problem leads to the study of parabolic equations. One of the classic assumptions of the Black-Scholes model solution (1973) is that the investor's portfolio revalues continuously. This dynamic implies transaction costs, due to the buying/selling of stocks for maintaining the portfolio's equilibrium. Black-Scholes models that include proportional transaction costs have been studied by many authors Leland (1985). In this work we suppose that transaction costs behave as a non-increasing positive linear function, $h(x) = a - bx$, ($a, b > 0$) which depends on the stock trading needed for hedging the portfolio that replicates the contingent claim. Moreover, if the underlying asset follows a jump-diffusion process, Merton (1993), we obtain an integro-differential evolution problem, with boundary value conditions, extending the paper Amster, Averbuj, Mariani, Rial (2005). Under adequate conditions, we extend the results obtained in ([2]), including a jump process, and we propose the existence of a unique convex solution to the corresponding evolution Dirichlet problem.

Keywords: Poisson-diffusion process; transaction costs; nonlinear partial differential equations; viscosity solution.

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1. Introduction

Since the Black and Scholes paper ([4]) presented a formula for pricing option, there has been an increasing interest in problems arising in Financial Mathematics and in particular in derivatives pricing. The standard approach to this problem leads to the study of parabolic equations.

In the standard Black-Scholes model, a basic assumption is that the stock price follows a geometric Brownian motion through time.

Empirical studies demonstrate the existence of systematic biases on stock price series. Several authors as Merton ([16]) propose an option pricing model that explicitly admits jumps in the underlying security return process; he considers a dynamic stock price with two components:

- a) A continuous component represented by a Wiener process;
- b) A jump component represented by a "Poisson-driven" process.

The stock price return can be formally written as:

$$\frac{dS}{S} = (\alpha - \lambda k)dt + \sigma dZ + (Y - 1)dq$$

where α is the instantaneous expected return on the stock; σ^2 is the instantaneous variance of the return, conditional on no new information arriving, λ is the mean number of arrivals per unit of time, $k = E(Y - 1)$, $Y > 0$ where $(Y - 1)$ is the random variable percentage change in the stock price and E is the expectation operator over the random variable Y and dq is a Poisson process with rate λt (for details see [16],[18]).

In [16],[18] an option pricing formula is derived from which the following partial integro-differential equation (PIDE) on variables τ and S is obtained:

$$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + (\alpha - \lambda k) S \frac{\partial F}{\partial S} - \frac{\partial F}{\partial \tau} - g(S, \tau) F + \lambda E \{F(SY, \tau) - F(S, \tau)\} = 0 \quad (1)$$

Here τ is time until the expiration of the option and $g(S, \tau)$ is the equilibrium, instantaneous rate of return on the option $F(S, \tau)$.

If the jump components represent non-systematic risk reflecting the arrival of new information specific to the firm or its industry, they will be uncorrelated with the market. The expect return on these securities must then be equal to the riskless rate; therefore, in (1) α and $g(S, \tau)$ will be changed by r (for details see [16],[18]).

In order to propose a model that fit well with empirical investigations, Kou ([13]) supposes that the logarithm of jump sizes has a double exponential distribution and that jump times correspond to the event times of a Poisson process. Press ([19]) supposes that the jump sizes have a Lognormal distribution. For other empirical papers see [4]-[6]-[12].

Even more, Merton ([18]) proposed a model where the jump sizes are normally distributed and Kim Jig-M Jung ([12]) supposed that jump sizes have a Poisson distribution.

Another classical supposition of the Black- Scholes model is that the investors portfolio revalues in a continuous way.

This dynamic implies transaction costs, due to the buying/selling of stocks for maintaining the portfolio's equilibrium. Black- Scholes models that include transaction costs have been studied by many authors ([20],[15], [11], [17]).

In this work we suppose that transaction costs behave as a non-increasing positive linear function, $h(x) = a - bx$, ($a, b > 0$), which depends on the stock trading needed for hedging the portfolio that replicates the contingent claim, and that the price $F(S, t)$ is always positive; for details ([21]).

Our main contribution is to extend the results of [11], [17] and [2] proving that under adequate conditions there exists a unique convex solution.

In the next section we study a generalization of PIDE (1) including transaction costs within the model.

2. Derivation of valuation problem

Just as Merton ([16]), we assume the stock price return follows the SDE:

$$\frac{dS}{S} = (\alpha - \lambda k)dt + \sigma dZ + dq$$

rewritten as

$$\begin{aligned} \frac{dS}{S} &= (\alpha - \lambda k)dt + \sigma dZ \quad \text{if the Poisson process does not occur} \\ &= (\alpha - \lambda k)dt + \sigma dZ + (Y - 1), \quad \text{if the Poisson process occur} \end{aligned}$$

dZ and dq are assumed to be independent, $k = E(Y - 1)$, where $(Y - 1)$ is the random variable percentage change in the stock price, dq is a Poisson process with rate λt

Following the idea of Leland ([15]), if the value of the option is denoted by $V(S, t)$, where S is the value of the underlying asset, we construct the portfolio $\Pi = V - \Delta S$.

We consider the change $\delta \Pi$ in Π over a discrete time-step δt , then we have:

$$\delta \Pi = \delta V - \Delta \delta S - [(a - b|v|)S|v|] = \delta V - \Delta \delta S - [(a - b|v|)\delta S|v|] - [(a - b|v|)S|v|]$$

Where V is the number of shares of the asset which are traded in order to keep the equilibrium of the portfolio in the period $(t, t + \delta t)$.

Using Ito's Lemma for the continuous part and analogous lemma for the jump part ([14]) we have

$$\delta V = \left[\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S(r - \lambda k) \frac{\partial V}{\partial S} \right] \delta t + \sigma \frac{\partial V}{\partial S} \delta Z + [V(Sy, t) - V(S, t)] \delta q$$

and

$$\delta V - \Delta \delta S = \left[\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S(r - \lambda k) \frac{\partial V}{\partial S} \right] \delta t$$

$$+ \sigma \frac{\partial V}{\partial S} \delta Z + [V(Sy, t) - V(S, t)] \delta q$$

$$- \Delta [S(r - \lambda k) \delta t + \sigma S \delta Z + Sy \delta q]$$

$$-\left[(a-b|v|)\delta S|v|\right]-\left[(a-b|v|)S|v|\right]$$

If we choose $\Delta = \frac{\partial V}{\partial S}(S, t)$ as the number of shares to maintain in the instant t , then the number of shares of the asset which are traded in the period $(t, t + \delta t)$ is:

$$v = \frac{\partial V}{\partial S}(S + \delta S, t + \delta t) - \frac{\partial V}{\partial S}(S, t)$$

We apply Taylor expansion to the right hand for small δt , δS and δq , it follows that

$$v \simeq \frac{\partial^2 V}{\partial S^2} \delta S + \left[\frac{\partial V}{\partial S}(Sy, t) - \frac{\partial V}{\partial S}(S, t) \right] \delta q + O(\delta t)$$

As $\delta S = \sigma S \phi \sqrt{\delta t} + O(\delta t)$, with, $\phi \sim N(0, 1)$, we took the expected value to the right hand, and leading order we obtain that

$$E \left\{ \left[\frac{\partial V}{\partial S}(Sy, t) - \frac{\partial V}{\partial S}(S, t) \right] \delta q \right\} = \lambda \delta t E \left\{ \left[\frac{\partial V}{\partial S}(Sy, t) - \frac{\partial V}{\partial S}(S, t) \right] \right\} \simeq O(\delta t)$$

and

$$v \simeq \frac{\partial^2 V}{\partial S^2}(S, t) \sigma S \phi \sqrt{\delta t}$$

Then,

$$\begin{aligned} E \left[(a-b|v|)S|v| \right] &= E \left[aS|v| - bSv^2 \right] = E(aS|v|) - E(bSv^2) = \\ &= aE \left(\left| \frac{\partial^2 V}{\partial S^2} \right| \sigma S^2 |\phi| \right) - bE \left(\left(\frac{\partial^2 V}{\partial S^2} \right)^2 S^3 \sigma^2 \phi^2 \right) \end{aligned} \quad (2)$$

Using that $E(|\phi|) = \sqrt{\frac{2\delta t}{\pi}}$, $E(\phi^2) = \delta t$ and (2) we find that

$$E \left[(a-b|v|)S|v| \right] = \left| \frac{\partial^2 V}{\partial S^2} \right| \sigma S^2 \sqrt{\frac{2}{\pi}} \sqrt{\delta t} a - bS^3 \left(\frac{\partial^2 V}{\partial S^2} \right)^2 \sigma^2 \delta t$$

As

$$E \{ \delta \Pi \} = E \{ \delta V - \Delta \delta S \} - E \left\{ (a-b|v|)S|v| \right\}$$

we obtain that $E \{ \delta \Pi \}$ is equal to

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S(r - \lambda k) \frac{\partial V}{\partial S} + \lambda E[V(Sy, t) - V(S, t)] - \left| \frac{\partial^2 V}{\partial S^2} \right| \delta S^2 \sqrt{\frac{2}{\pi \delta t}} a - bS^3 \left(\frac{\partial^2 V}{\partial S^2} \right)^2 \sigma^2 \right) \delta t$$

In the presence of the jump process dq , the return on the portfolio will not be riskless, but if the jump component of the stocks return represents non-systematic risk, then the jump component will be uncorrelated to the market, if the CAPM holds, the expect return must equal the riskless rate, for details see ([16]), ([18]).

Hence we obtain the equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - a \left| \frac{\partial^2 V}{\partial S^2} \right| \sigma S^2 \sqrt{\frac{2}{\pi \delta t}} + \left(\frac{\partial^2 V}{\partial S^2} \right)^2 b S^3 \sigma^2 + (r - \lambda k) S \frac{\partial V}{\partial S} - rV + \lambda E[V(Sy, t) - V(S, t)] = 0 \quad (3)$$

Assuming that a is small enough, we have that $\tilde{\sigma}^2 = \sigma^2 \left(1 - \frac{a}{\sigma} \sqrt{\frac{2}{\pi \delta t}} \right) > 0$

We know that the classical solution of the Black-Scholes equation is

$$C(S, t) = SN(d_1) - Ke^{-r(T-t)} N(d_2)$$

and

$$\frac{\partial C}{\partial S}(S, t) = N(d_1)$$

$$\frac{\partial^2 C}{\partial S^2}(S, t) = \frac{e^{-\frac{d_1^2}{2}}}{S\sigma\sqrt{t}} > 0$$

Because the european call option is a convex function with respect to S , we assume $\frac{\partial^2 V}{\partial S^2} > 0$.

In the next section we study the problem (3) under the Dirichlet boundary conditions.

2.1. Solutions of the evolution problem

In this section we study the nonstationary problem (3) (see [8]-[9] for details) under initial-Dirichlet conditions, namely

$$\frac{\partial V}{\partial t} + \frac{1}{2} \tilde{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + b\sigma^2 S^3 \left(\frac{\partial^2 V}{\partial S^2} \right)^2 + r \left(\frac{\partial V}{\partial S} S - V \right) + \lambda \int (V(Sy) - V(S)) g(y) dy = 0$$

$$\text{With } \begin{cases} V(T, S) = f(S) & S \in (c, d), \\ V(t, c) = f(c) \\ V(t, d) = f(d). \end{cases}$$

Let us introduce the change of variables given by

$$w(t, x) = V(T-t, e^x) \text{ and } Z(t, x) = w_{xx} - w_x \text{ in the domain } (0, T) \times (\bar{c}, \bar{d})$$

So that the equation is transformed into:

$$-Z_t + a(x, Z, p)Z_{xx} + d(x, Z, p) + \lambda J(Z) = 0$$

$$Z(0, x) = Z_0(x)$$

$$Z(t, \bar{c}) = Z_0(\bar{c}), Z(t, \bar{d}) = Z_0(\bar{d})$$

With

$$a(x, Z, p) = \frac{1}{2} \tilde{\sigma}^2 + 2b\sigma^2 e^{-x} Z,$$

$$p = Z_x$$

$$d(x, Z, p) = 2b\sigma^2 e^{-x} Z^2 - 6b\sigma^2 e^{-x} pZ + 2b\sigma^2 e^{-x} p^2 - rZ + \left(r - \lambda k - \frac{1}{2} \tilde{\sigma}^2 \right) p$$

$$J(Z) = \int (Z(t, x + 1ny) - Z(t, x)) g(y) dy$$

$$\text{and } Z_0(x) = f''(e^x) e^{2x}$$

Equivalently, we have

$$Z_t + F(t, x, Z, p, Z_{xx}) - \lambda J(Z) = 0 \quad (4)$$

under initial-Dirichlet conditions

$$Z(0, x) = Z_0(x) \quad x \in [\bar{c}, \bar{d}], \quad (5)$$

$$Z(t, \bar{c}) = Z_0(\bar{c}), Z(t, \bar{d}) = Z_0(\bar{d})$$

where

$$F(x, t, Z, p, Z_{xx}) = -a(x, Z)Z_{xx} - d(x, Z, p).$$

Let us define $\tilde{a}(x, Z) = a(x, [Z]_+)$ and $\tilde{d}(x, Z, p) = d(x, Z, p)$.

We consider the problem

$$Z_t + \tilde{F}(t, x, Z, p, Z_{xx}) - \lambda J(Z) = 0 \quad (6)$$

under the condition (5) where $\tilde{F}(t, x, Z, p, Z_{xx}) = -\tilde{a}(x, Z)Z_{xx} - \tilde{d}(x, Z, p)$

2.2. Previous definitions

Definition 1 (see [7]) Given a function Z , we shall say that $Z_t + F(t, x, Z, p, Z_{xx}) - \lambda J(Z) \leq 0$ (resp., ≥ 0) in viscosity sense for $(t, x) \in (0, T) \times [\bar{c}, \bar{d}]$ if one of the following equivalent items holds:

(1) $\tau + F(t, x, Z, p, X) - \lambda J(Z) \leq 0$ (resp., ≥ 0) for all (τ, p, X) in $P^+Z(t, x)$ (resp., $P^-Z(t, x)$);

(2) $\varphi_t + F(t, x, Z, \varphi_x, \varphi_{xx}) - \lambda J(Z) \leq 0$ (resp., ≥ 0) in the classical sense, for each smooth $\varphi \in C^2([0, T] \times [\bar{c}, \bar{d}])$ such that $Z - \varphi$ has a local maximum (resp., minimum) at (t, x)

(3) $\varphi_t + F(t, x, \varphi, \varphi_x, \varphi_{xx}) - \lambda J(\varphi) \leq 0$ (resp., ≥ 0) in the classical sense, for each smooth $\varphi \in C^2([0, T] \times [\bar{c}, \bar{d}])$ such that $Z - \varphi$ has a global strict maximum (resp., minimum) at (t, x) .

Where the parabolic semijets $P^\pm$ are defined as

$$P^\pm Z(t, x) = \left\{ \begin{array}{l} (\tau, p, X) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : \\ Z(s, y) \leq (\text{resp., } \geq) Z(t, x) + \tau(t-s) + p(x-y) + \frac{1}{2}X(x-y)^2 \\ + o(|t-s| + |x-y|^2) \text{ as } (s, y) \rightarrow (t, x) \end{array} \right\}$$

Definition 2 An upper (resp., lower) semicontinuous Z on $[0, T] \times [\bar{c}, \bar{d}]$ is a subsolution (resp., a supersolution) to the problem (5)-(6) if $Z_t + F(t, x, Z, p, Z_{xx}) - \lambda J(Z) \leq 0$ (resp., ≥ 0) in viscosity sense at all $(t, x) \in [0, T] \times [\bar{c}, \bar{d}]$, $Z(0, x) \leq Z_0(x)$

(resp., $\geq$) at all $x \in [\bar{c}, \bar{d}]$ and $Z(t, \bar{c}) \leq Z_0(\bar{c})$, $Z(t, \bar{d}) \leq Z_0(\bar{d})$ (resp., $\geq$) for $0 \leq t < T$

Definition 3 Z^* is the Upper semicontinuous envelope of Z

2.3. Existence and uniqueness of viscosity solution

Theorem 4 (Comparison principle) Let a bounded $u \in USC([0, T] \times [\bar{c}, \bar{d}])$ (resp. bounded $v \in LSC([0, T] \times [\bar{c}, \bar{d}])$) be a subsolution (resp. supersolution) of (6) and there exists $C > 0$ such that $|u(x) - v(y)| \leq C(1 + |x| + |y|)$ then $u \leq v$ on $[0, T] \times [\bar{c}, \bar{d}]$

Before doing the proof, we recall a lemma used by ([10]) a proposition used by ([3]) and ([1])

Lemma 5 ([10], Theorem 3). Let $g \in C([0, T])$ be such that

$$g(0) = 0, \int_0^1 \frac{ds}{g(s)} = +\infty.$$

Let $f \in USC([0, T])$ be nonnegative, bounded from above and suppose that

$$\min(f'(t) - g(f(t)), 0) \leq 0 \text{ for all } t \in [0, T] \quad (7)$$

in viscosity sense, then $f \equiv 0$.

Proposition 6 ([3], proposition 1). Let $F \in C([0, T] \times [\bar{c}, \bar{d}] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ be elliptic. Assume that there are $\underline{u}, \bar{v} \in C([0, T] \times [\bar{c}, \bar{d}])$ respectively a viscosity subsolution and a supersolution of (6) such that $\underline{u} \leq \bar{v}$ on $[0, T] \times [\bar{c}, \bar{d}]$. Assume also that the following principle of comparison holds: if $u \in USC([0, T] \times [\bar{c}, \bar{d}])$ be a subsolution and $v \in LSC([0, T] \times [\bar{c}, \bar{d}])$ be a supersolution) of (6) with $u \leq v$ and $v \geq \underline{u}$ then $u \leq v$ on $[0, T] \times [\bar{c}, \bar{d}]$.

Then there is an unique viscosity solution $u \in C([0, T] \times [\bar{c}, \bar{d}])$ such that $\underline{u} \leq u \leq \bar{v}$ on $[0, T] \times [\bar{c}, \bar{d}]$

Proposition 7 ([1], theorem 3.4) Let $\underline{u}, \bar{v}$ satisfying $e^{-n|x|}\underline{u}, e^{-n|x|}\bar{v} \in L^\infty([0, T], L^\infty(\bar{c}, \bar{d}))$ be respectively a sub/super solution of (6)

Then $\underline{u} \leq \bar{v}$.

Proof of theorem 4. Let $\theta(t) = \sup_{[\bar{c}, \bar{d}]} (u(t, x) - v(t, x))$. We want to prove that $\theta \equiv 0$. Using the previous lemma, it suffices to show that θ^* fulfills (7) for

$$g(\theta^*) = \theta^*(K + r)$$

Let $t_0 \in [0, T]$, with $\theta^*(t_0) > 0$ and take $\varphi \in C^1([0, T])$ such that $\theta^*(t_0) - \varphi(t_0) = 0$ is a strict global maximum.

For fixed $\delta > 0$ small enough, we set the function $\Psi_\delta(t, x) = u(t, x) - v(t, x) - \frac{\delta}{2}|x|^2 - \varphi(t)$ has a global maximum point $(t_\delta, x_\delta) \in [0, T] \times [\bar{c}, \bar{d}]$. Similar to the proof of ([1], theorem 3.4) we can choose a sequence, such that

$$\lim_{\delta \rightarrow 0} \frac{\delta}{2} |x_\delta|^2 = 0, \lim_{\delta \rightarrow 0} t_\delta = t_0, \lim_{\delta \rightarrow 0} u(t_\delta, x_\delta) - v(t_\delta, x_\delta) = \theta^*(t_0) \quad (8)$$

And as a consequence

$$\lim_{\delta \rightarrow 0} \theta(t_\delta) = \theta^*(t_0)$$

We may suppose that $t_\delta > 0$ for all δ , otherwise we should have $0 < \theta^*(t_0) = \lim_{\delta \rightarrow 0} (u(0, x_\delta) - v(0, x_\delta)) \leq Z_0(x_\delta) - Z_0(x_\delta) = 0$.

Choose $\varepsilon > 0$, for any fixed $\delta > 0$ and set $\Psi_{\varepsilon\delta}(t, x, y) = u(t, x) - v(t, y) - \frac{\delta}{2}|x|^2 - \frac{|x-y|^2}{2\varepsilon} - \varphi(t)$. Following the general viscosity solution technique, we consider a global maximum point $(t_{\varepsilon\delta}, x_{\varepsilon\delta}, y_{\varepsilon\delta}) \in (0, T] \times [\bar{c}, \bar{d}] \times [\bar{c}, \bar{d}]$ of $\Psi_{\varepsilon\delta}$. Then by the proof of ([1], theorem 3.4) we have

$$\lim_{\varepsilon \rightarrow 0} \frac{|x_{\varepsilon\delta} - y_{\varepsilon\delta}|^2}{2\varepsilon} = 0, \lim_{\varepsilon \rightarrow 0} t_{\varepsilon\delta} = t_\delta, \lim_{\varepsilon \rightarrow 0} u(t_{\varepsilon\delta}, x_{\varepsilon\delta}) - v(t_{\varepsilon\delta}, y_{\varepsilon\delta}) = u(t_\delta, x_\delta) - v(t_\delta, x_\delta)$$

In view of (8) we assume that $t_{\varepsilon\delta} > 0$ and $u(t_{\varepsilon\delta}, x_{\varepsilon\delta}) - v(t_{\varepsilon\delta}, y_{\varepsilon\delta}) > 0$ for $\varepsilon > 0$ small enough

Theorem 8.3 of ([7]) provides $\left(p_{\varepsilon\delta}, \frac{(x_{\varepsilon\delta} - y_{\varepsilon\delta})}{\varepsilon}, A_{\varepsilon\delta}, B_{\varepsilon\delta} \right)$ such that $(p_{\varepsilon\delta} + \varphi'(t_{\varepsilon\delta}), \frac{(x_{\varepsilon\delta} - y_{\varepsilon\delta})}{\varepsilon} + 2\delta A_{\varepsilon\delta} + \delta) \in P^+ u(t_{\varepsilon\delta}, x_{\varepsilon\delta})$,

$$(p_{\varepsilon\delta}, \frac{(x_{\varepsilon\delta} - y_{\varepsilon\delta})}{\varepsilon}, B_{\varepsilon\delta}) \in P^- v(t_{\varepsilon\delta}, y_{\varepsilon\delta})$$

$$\text{and } \begin{pmatrix} \varepsilon & 0 \\ 0 & -B_{\varepsilon\delta} \end{pmatrix} \leq \frac{3}{\varepsilon} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (9)$$

Because u and v are respectively a subsolution and supersolution of (6), we have (omitted the dependence on ε and δ)

$$p + \varphi'(t) - \tilde{\alpha}(x, u)(A + \delta) - \tilde{d}(x, u, \frac{(x-y)}{\varepsilon} + 2\delta) - \lambda J(u) \leq 0$$

$$p - \tilde{\alpha}(y, v)B - \tilde{d}(y, v, \frac{(x-y)}{\varepsilon}) - \lambda J(v) \geq 0$$

Subtract two inequalities, and take the upper limit as $\varepsilon \rightarrow 0$.

$$\begin{aligned} \varphi'(t_\delta) &\leq \limsup_{\varepsilon \rightarrow 0} (\tilde{\alpha}(x, u(t_\delta, x))(A + \delta) - \tilde{\alpha}(y, v(t_\delta, y))B + \tilde{d}(x, u, \frac{(x-y)}{\varepsilon} + \delta x) \\ &\quad - \tilde{d}(y, v, \frac{(x-y)}{\varepsilon}))\lambda \int |(u(t_\delta, x_\delta + 1nz) - u(t_\delta, x_\delta) - v(t_\delta, x_\delta + 1nz) + v(t_\delta, x_\delta))| g(z) dz \end{aligned}$$

$$\leq \lim_{\varepsilon \rightarrow 0} \sup \left(\tilde{F} \left(y, v, \frac{(x-y)}{\varepsilon}, B \right) - \tilde{F} \left(y, u, \frac{(x-y)}{\varepsilon}, B \right) + \lim_{\varepsilon \rightarrow 0} \sup \left(\tilde{F} \left(y, u, \frac{(x-y)}{\varepsilon}, B \right) - \tilde{F} \left(y, u, \frac{(x-y)}{\varepsilon}, A \right) \right. \right. \\ \left. \left. + \lim_{\varepsilon \rightarrow 0} \sup \left(\tilde{F} \left(y, u, \frac{(x-y)}{\varepsilon}, A \right) - \tilde{F} \left(x, u, \frac{(x-y)}{\varepsilon} \right) + \delta x, A + \delta \right) \right)$$

Because $\tilde{F}$ be elliptic and (9), the second sumand is lower than zero.

After some computations, we have

$$\varphi'(t_\delta) \leq (K+r) \vartheta(t_\delta) + \tilde{\delta} \alpha(x_\delta, u(t_\delta, x_\delta)) \quad (10)$$

Using (8) taking limit $\delta \rightarrow 0$ in (10) we conclude that $\varphi'(t_0) \leq (K+r) \vartheta(t_0)$ and this is a contradiction because using lemma 5. we have $\vartheta(t) \equiv 0$

Theorem 8 Let $Z_0 \in C(\bar{c}, \bar{d})$. Assume that there are $\underline{u}, \bar{u} \in C([0, T] \times [\bar{c}, \bar{d}])$ respectively a viscosity subsolution and a supersolution of (6) then there is a unique viscosity solution $Z \in C([0, T] \times [\bar{c}, \bar{d}])$ to (6)-(5) such that $\underline{u} \leq Z \leq \bar{u}$

— **Proof.** Let $\underline{u} = 0$ is a subsolution and a simple computation shows $\underline{u}(t, x) = e^{\beta t} (At + Be^x)$ is a supersolution of (6) for A big enough and B nonnegative and small.

$\tilde{F} \in C([0, T] \times [\bar{c}, \bar{d}] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ and for $A \leq B, A \leq B$,

$$\tilde{F}(t, x, Z, p, A) = -\tilde{\alpha}(x, Z, p)A - \tilde{d}(x, Z, p) \geq -\tilde{\alpha}(x, Z, p)B - \tilde{d}(x, Z, p) = \tilde{F}(t, x, Z, p, B)$$

then $\tilde{F}$ be elliptic. By proposition 6. and theorem 4. the proof follows.

Remark 9 By the previous theorem and the comparison principle we give $0 \leq Z(t, x) \leq e^{\beta t} (At + Be^x)$, then $\tilde{a}(x, Z) = a(x, Z)$, and Z is unique viscosity solution of (4)-(5)

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