Balanced Coalitional Matchings

Abstract: In this note we define a balancedness property for a particular two sided many-to-many matching problem developed by Dimitrov and Lazarova (2008), and prove that this is a necessary and sufficient condition for the existence of stable coalitional matchings, the solution concept proposed for this model in the latter reference. The proof is carried out by using some results from the theory of hedonic games. We also show that hedonic games defines an equivalence relation over the set of matching problems.

Keywords: Man-to-many matching problems; Coalitional matchings; hedonic games.

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Juan Carlos Cesco*

* a Instituto de Matemática Aplicada San Luis (UNSL-CONICET) Av. Ejército de los Andes 950, 5700 San Luis, Argentina(jcesco@unsl.edu.ar) and b Departamento de Matemática (U.N. San Luis) Chacabuco y Pedernera, 5700 San Luis, Argentina Corresponding author: Juan Carlos Cesco Instituto de Matemática Aplicada San Luis Av. Ejército de los Andes 950 D5700HHW San Luis, ARGENTINA e-mail address: jcesco@unsl.edu.ar Phone/Fax #: 54 2652 422803

1. Introduction

In a recent paper, Dimitrov and Lazarova (2008) introduced a two sided market model which relates coalition from one side of the market to coalition from the other side. It has several distinguishing features that make it a very interesting model. First, it is a special kind of many-to-many matching problem which includes preferences depending on the colleagues (Dutta and Massó (1997), Echenique and Yenmez (2007)) in both sides of the market. Second, the model is not only a matching model but also a coalition formation model and its main solutions, coalitional matchings, reflect both point of view in an integrated way. The paper focuses mainly on the existence of stable coalitional matchings and the results have been obtained under the assumption that individual overall preferences on coalitions containing agents of both sides of the market are derived from two individual basic preferences on coalitions of each side of the market separately according to the classical many-to-many matching theory (i.e. Echenique and Oviedo (2006-b)). Several lexicographical proposals to derive overall preferences from the basic ones are presented giving rise to different existence results.

In this short note we adopt a different approach to state a general existence theorem. First we state a balancedness condition on a matching problem and then we show that this is a necessary and sufficient condition for the existence of coalitional matchings. A major depart from the point of view adopted in Dimitrov and Lazarova (2008) is that our condition focus mainly on the preference profile of the overall preferences rather than on the basic individual preferences. This allows us to relate the two sided market model with a hedonic game (Banerjee et al. (2001), Bogolmolnaia and Jackson (2002)) in a very direct way and then, to use results from this theory to get our existence theorem.

The paper is organized as follows. In the next section we introduce the model and the notion of coalitional matching. In Section 3 we state a couple of balancedness properties on matching problems, while Section 4 is devoted to prove our main result. Section 5 present an intermediate notion of solution which resembles the stability* concept developed in Echenique and Oviedo (2006-a) and Echenique and Yenmez (2007) which could be used to check easier the core stability of a coalitional matching. We close the note stressing on the interrelationships between coalitional matching problems and hedonic games. In particular we show that both models are strongly related in the sense that each hedonic games define an equivalence class on the set of coalitional matching problems. We also outline some other lines of work on the subject we are carring on nowadays.

2. Coalitional matchings

A coalitional many-to-many matching problem, shortly a matching problem, consists of two disjoint finite sets of agents, the set R of 'researchers' and the set S of 'students'. For any subset A of $R\cup S$, and $t\in R\cup S$, $A(t)$ will stand for the family of subsets of A containing s. In some cases, $A(t)$ may be an empty family. The model also assumes that each element $t\in R\cup S$ is endowed with an individual preference $\underline{>}_t$ over $(R\cup S)(t)$ which is strict, complete and transitive on its domain. We recall that a preference $\underline{>}_t$ is strict if $x\underline{>}y$ and $x\neq y$ implies that $y\underline{>}x$ does not hold. We will use $>$ to denote the strict preference derived from $\underline{>}$. Given a non-empty subset A, 2^A will stand for the family of subsets of A.

We denote a matching problem by $(R,S;\underline{>})$, where $\geq=(\geq_t)_{t\in R\cup S}$ is the preference profile. As a pre-solution concept we consider the set of all coalitional matchings (Dimitrov and Lazarova (2008)). A coalitional matching in $(R,S;\underline{>})$ is a function $\mu{:}R\cup S\to 2^{R\cup S}$ satisfying:

a) For all $t \in R \cup S, t \in \mu(t)$.
b) If $\mu(t)=A$ for some $t \in R \cup S$, then $\mu(t')=A$ for all $t' \in A$.

For each $r \in R, \mu(r)$ is a group of researchers or teaching team, including himself, along with a group of students to whom the teaching team wants to teach. Similarly, for each $s \in S$, $\mu(s)$ consists of a group of students including himself, and a group of researchers from whom they want to learn. In this model the coalitions, namely, the subsets of $R \cup S$, are named, according to Dimitrov and Lazarova (2008), universities, although they can be very atypical. For instance, a university can have no student or no researcher either. Given a coalitional matching μ, the family $\{A \subseteq R \cup S : A = \mu(t)$ for at least one $t \in R \cup S\}$ is a partition of $R \cup S$ (Dimitrov and Lazarova (2008)) denoted by π^{μ} or, alternatively, by $(\mu(t))_{t \in R \cup S}$.

A pair (A, μ'), where $A \subseteq R \cup S$ is a non-empty coalition and μ' is a coalitional matching, blocks a coalitional matching μ if
a) For all $t \in A, \mu'(t) \in A(t)$
b) For all $t \in A, \mu'(t) >_t \mu(t)$

A coalitional matching μ is core stable, or simply stable, if it cannot be blocked. We use $\mathcal{C}(R,S; \geq)$ to denote the set of all stable matchings. As usual, one of the main issues regarding this model is to decide whether $\mathcal{C}(R,S; \geq)$ is empty or not.

A coalitional matching is individually stable (Dutta and Massó (1997)) if there is no $t \in R \cup S$ such that $\{t\} >_t \mu(t)$. Clearly, each core stable matching is individually stable. Otherwise $(\{t\}\ \mu')$, where μ is the coalitional matching assigning $\mu'(t)=\{t\}$ to each $t \in R \cup S$ would be an individual block for μ.

In the aforementioned reference Dimitrov and Lazarova (2008), each individual preference $\geq_t$ is derived from two basic individual preferences, $\geq_t^T$ and $\geq_t^G$ The preference $\geq_t^T$ orders the family of researchers teams in $R(t)$ if $t \in R$ or in 2^R if $t \in S$, while $\geq_t^G$ orders the family of students groups in 2^S if $t \in R$ or in $S(t)$ if $t \in S$. According to how the overall preference $\geq_t$ is gathered from $\geq_t^T$ and $\geq_t^G$ for each $t \in R \cup S$, several conditions indicating the non-emptiness of $\mathcal{C}(R,S; \geq)$ are provided.

Here we concentrate directly on the overall preference profile and state a balancedness property which turns to be necessary and sufficient for the non-emptiness of $\mathcal{C}(R,S; \geq)$.

3. Balanced matching problems

We start this section by recalling the notions of distribution and that of balanced family of coalitions with respect to a distribution due to Iehlé (2007).

Let $N=\{1,...,n\}$ be a finite set and $b=(b^A)_{\Phi \neq A \in 2^N}$ be a family of non-negative vectors in R^N satisfying:

$$b_i^A = 0 \text{ for all } i \notin A,$$

and

$$b_i^N > 0 \text{ for all } i \in N.$$

A non-empty subfamily $\mathcal{B} \subseteq 2^N$ is b-balanced (Billera (1970)) if there exist positive weights $\lambda=(\lambda_A)_{A \in \mathcal{B}}$ such that

$$\mathcal{B} \qquad \sum_{B \in \mathrm{B}} \lambda_B b^B = b^N$$

In the case that for each non-empty coalition $A \in 2^N, b^A = \chi^A$, b-balancedness coincides with the classical notion of balancedness (see Shapley (1967)). Here χT is the n-dimensional indicator vector of a coalition $A \subseteq N$, namely, $\chi^{A,i=1}$ if $i \in A$ and 0 otherwise.

A family $I=(I(A))_{A\in 2^N}$ is called a distribution in N (Iehlé (2007)) if, for each non-empty coalition $A\in 2^N, \phi \neq I(A) \subseteq A$. Given a distribution I, a family $\mathcal{B}\subseteq 2^N$ is I -balanced if it is b-balanced with respect to the family of vectors $b=(\chi^{I(A)})_{\varphi\neq A\in 2^N}$ namely, if

$$\sum_{B\in \mathrm{B}} \lambda_B \chi^{I(B)} = b^N.$$

A matching problem $(R,S;\geq)$ is p-balanced if there exists a distribution I in $R\cup S$ such that for each I -balanced subfamily $\mathcal{B}\subseteq 2^{R\cup S}$ there exists a coalitional matching μ such that for all $t\in R\cup T$

$$\mu(t) \geq_t B,$$

for at least one $B\in\mathcal{B}(t)$. Here $\mathcal{B}(t)$ denotes the coalitions in $\mathcal{B}$ containing t.

Our definition of p-balancedness for matching problems is strongly motivated for that of pivotal balancedness for hedonic games due to Iehlé (2007). In the case that the distribution $I=(A)_{\varphi\neq A\in 2^N}$, a p-balanced matching problem will be called only balanced, and like in the case of p-balancedness, this definition is strongly related to that of ordinal balancedness of Bogolmolnaia and Jackson (2002) for hedonic games. Both conditions have already been use (Cesco (2008)) in the framework of many-to-one matching problems to provide a necessary and sufficient condition for the existence of stable* matchings (Echenique and Oviedo (2006-a)). The following example shows that balancedness is a strictly stronger condition than p-balancedness.

Example

Let $(R,S;\underline{>})$ be a matching problem with $R=\{1\},S=\{2,3\}$ and the individual preferences given by:

$$\{1,2\} >_1 \{1,3\} >_1 \{1.2.3\} >_1 \{1\}$$

$$\{1,2\} >_2 \{2,3\} >_2 \{1.2.3\} >_2 \{2\}$$

and

$$\{1,3\} >_3 \{2,3\} >_3 \{1.2.3\} >_3 \{3\}$$

The corresponding associated hedonic game $(R\cup S;\bar{\geq})$ is one studied in Banerjee et al. (2001) and later on, in Iehlé (2007). In this latter reference it is proven that this game is pivotally balanced but not balanced, having a unique core partition $\pi^*=(\{1.2\},\{3\})$. Consequently, the matching problem $(R,S;\geq)$ turns to be p-balanced although it is not balanced. Moreover, it is easy to see that $(R,S;\geq)$ is p-balanced with respect to the distribution I given by: $I(\{i\})=\{i\},i=1,2,3$, $I(\{1,2\})=\{1,2\},I(\{1,3\})=\{1\},I(\{2,3\})=\{2\}$, and $I(\{1,2,3\})=\{1,2\}$.

Remark 1 We point out that the preference profile of the example does not belong to any of the domains $\mathcal{D}_2$, $\mathcal{D}_4$ or $\mathcal{D}_6$ introduced in Dimitrov and Lazarova (2008).

4. Main characterization result

In this section we prove the following characterization result.

Theorem 1 Let $(R,S;\underline{>})$ be a matching problem. Then, $C(R,S;\underline{>})\neq\varphi$ if and only if $(R,S;\underline{>})$ is p-balanced.

Proof The proof is carried out in two steps. First, we associate to $(R,S;\underline{>})$ a hedonic game (Banerjee et al. (2001), Bogolmolnaia and Jackson (2002)) and show an equivalence between core stable matchings in the matching problem and core-partitions in the game. Later we prove that p-balancedness in $(R,S;\underline{>})$ is equivalent to pivotal balancedness in the game, which is a necessary and sufficient condition for the non-emptiness of its set of core-partitions (Iehlé 2007).

Given $(R,S;\geq)$, let $(N;\geq)$ be an associated hedonic game where $N=R\cup S$ and $\bar{\geq}=(\bar{\geq}_t)_{t\in N}$ is a preference profile, where, for each $t\in N, \bar{\geq}_t$ is an individual preference defined on $N(t)$ as follows:

$$A\bar{\geq}_t B \text{ if and only if } A\geq_t B.$$

We recall that a partition $\pi=(\pi_1,\ldots,\pi_m)$ of N is a core-partition in $(N;\geq)$ if there is no subset $A\neq\varphi$ of N such that, for all $t\in A$

$$A\bar{\geq}_t\pi(t)$$

and

$$A\bar{\geq}_t\pi(t)$$

for at least one $t\in A$, where $\pi(t)$ represents the unique coalition in π containing player t. In our model of game, individual preferences in $\bar{\geq}$ are strict, and thus, π is a core-partition if and only if there is no $\Phi\neq A\subseteq N$ such that, for all $t\in A$

$$A\bar{\geq}_t\pi(t)$$

Now, let μ be a core stable matching in $(R,S;\geq)$. We claim that the associated partition π^{μ} belongs is a core-partition in $(N;\bar{\geq})$. In fact, if it were not the case, there would be $\Phi\neq A\subseteq N$ such that

$$A\bar{\geq}_t\pi(t)$$

for all $t\in A$. But then, the pair (A,μ') where μ' is the coalitional matching define by

$$\mu'(t)=A,$$

for all $t\in A$ and

$$\mu'(t)=N\backslash A$$

for all $t\in N\backslash A$ would block μ in contradiction to the assumption on this matching.

Conversely, if π is a core-partition in $(N;\bar{\geq})$, we claim that $\mu^{\pi}=(\mu(t))_{t\in R\cup S}$ is a stable matching in $(R,S;\geq)$. If it were not the case, there would be a pair (A,μ') blocking it. We first note that $\cup_{t\in A}\mu'(t)=A$ since for all $t\in A, \mu'(t)\subseteq A,$ and thus, the inclusion $\cup_{t\in A}\mu'(t)\subseteq A$ holds, while the reverse inclusion is also true due the fact that, for all $t\in A, t\in\mu'(t)$. From this we conclude that $(\mu'(t))_{t\in A}$ is a partition of A. Now, by letting $B=\mu'(t)$ for some $t\in A$, and taking into account that $\mu'(t)\succ_{t'}\mu^{\pi}(t)$ for all $t'\in\mu'(t)=B$ we could obtain that B objects π in $(N;\succsim)$, in contradiction with its choice.

Our result now follows from Theorem 2 of Iehlé (2007) which asserts that a hedonic game admits core-partitions if and only if it is pivotally balanced.

Remark 2 A partition π is individually rational in a hedonic game $(N;\bar{\geq})$ if there is no agent $t\in N$ such that $\{t\}\bar{>}_t\pi(t)$. A core partition is clearly individually rational, and thus, the mapping $\mu\rightarrow\pi^{\mu}$ $(\pi\rightarrow\mu^{\pi})$ sends individually stable matchings (individually rational partitions) onto individually rational partitions (individually stable matchings partitions) when its domain is the set of stable matchings (core-partitions). However this is also true even if this domain restriction is not present.

Corollary 2 A sufficient condition for $C(R,S;\geq)\neq\varphi$ is that $(R,S;\geq)$ is balanced.

Proof It follows from the fact that $(R,S;\geq)$ is balanced if and only if the associated game $(N;\bar{\succeq})$ is ordinally balanced and that ordinal balancedness implies pivotal balancedness for a hedonic game.

5. An intermediate notion of stability

The relationships between matching problems and hedonic games which have been used in this note to get some existence results have also been used successfully in the framework of many-to-one matchings problems where the preferences of the agents of one side of the markets may or not, depend on their colleagues (Cesco (2009), Cesco (2008)). In those paper, an intermediate notion of stability due to Echenique and Oviedo (2006-a) and Echenique and Yenmez (2007), named stability*, played a key role to prove the main existence results as soon as it is shown, in the latter two references, that the set of stable* matchings coincides with the set of core stable matchings, and in some cases, when substitutability is present, with the set of stable matchings. Here we adapt this, at a first glance, weaker notion of stability, to the framework of coalitional matching problems and prove that the set of coalitional matchings which are stable in this sense coincides, in fact, with the set of core stable matchings. Although there are some differences between the definition stated in Echenique and Yenmez (2007) and our, we keep the same name.

Let $(R,S;\geq)$ be a coalitional matching problem. A pair (t,B) blocks* a coalitional matching μ if

a) $t\in B\subseteq R\cup S$.

b) $B\cap\mu(t)=\{t\}$.

c) There is $C\subseteq\mu(t)$ such that $B\cup C\succ_{t'}\mu(t')$ for all $t'\in B\cup C$.

A coalitional matching μ is stable* if it is blocked* by no pair (t,B). We use $\mathbb{S}^*(R,S;\geq)$ to denote the set of all stable* matchings in $(R,S;\geq)$.

Lemma 3 A stable* coalitional matching μ in $(R,S;\geq)$ is individually stable.

Proof If it were not the case, there would be $t\in R\cup S$ such that $\{t\}\succ_t\mu(t)$. But then $(t,\{t\})$ would block⋆ μ, a contradiction.

Theorem 4 Let $(R,S;\geq)$ be a matching problem. Then, $\mathcal{C}(R,S;\geq)=\mathbb{S}^*(R,S;\geq)$.

Proof First we shall prove that $\mathcal{C}(R,S;\geq)\subseteq\mathbb{S}^*(R,S;\geq)$. To this end, let us assume that a coalitional matching μ does not belong to $\mathbb{S}^*(R,S;\geq)$. Then there is a pair (t,B) blocking* it. Let C be the subset of $\mu(t)$ associated to this blocking. Now let $A=B\cup C$ and μ' the matching assigning $\mu'(t')=B\cup C$ to each $t'\in B\cup C$, and $\mu'(t')=(R\cup S)\setminus(B\cup C)$ to each $t'\notin B\cup C$ Clearly μ' is a coalitional matching and the pair (A,μ') blocks μ. So μ is not core stable either.

To see the reverse inclusion, let $\mu\in\mathbb{S}^*(R,S;\geq)\setminus\mathcal{C}(R,S;\geq)$, and assume that (A,μ') is a block to μ. As we saw during the proof of Theorem 1, $(\mu'(t))_{t\in A}$ is a partition of A. Let $t\in A$. Since $\mu'(t)>_t\mu(t), \mu'(t)\neq\mu(t)$. We claim that (t,B), with $B=\{t\}\cup\mu'(t)\setminus\mu(t)$ $B=\{t\}\cup\mu'(t)\setminus\mu(t)$ is a block* to μ with $C=\mu'(t)\cap\mu(t)$. Indeed, $B\cap\mu(t)=\{t\}$, and since $B\cup C=\mu't$, $B\cup C>_t\mu(t)$. On the other hand, since μ' is a coalitional matching, $\mu'(t')=\mu'(t)$ for all $t'\in\mu'(t)\subseteq A$. From this and the fact that (A,μ') blocks μ, we finally get that

$$B\cup C=\mu'(t)$$

$$=\mu'(t')>_t\mu(t)$$

for all $t'\in B\cup C,$ proving our claim. This is a contradiction showing the inclusion $\mathbb{S}^*(R,S;\geq)\subseteq\mathcal{C}(R,S;\geq)$.

Theorem 4 states that, in order to detect if a matching μ is core stable, it is enough to verify if it is resistant to blocks supported by coalitions intersecting at least one member in the partition $(\mu(t))_{t\in R\cup S}$, in exactly one element.

6. Final remarks

We would like to stress on the relationships between coalitional matching problems and hedonic games. We have already shown how to associate a hedonic game to a given coalitional matching problem. But, as the example developed at the end of Section 3 suggests, it is possible to relate several coalitional matching problems $(R,S;\underline{\geq})$ to a given hedonic game $(N;\ \bar{\succeq}\ )$ by taking R and S as a non-trivial partition of N and defining $\bar{\succeq}_t = \geq_t$ for all $t\in R\cup S$. Thus, an equivalent relation over the set of coalitional matchings emerges in an obvious way: $(R,S;\underline{\geq})$ and $(R',S';\ \geq')$ are equivalent if and only if their associated hedonic game is the same.

Although the notion of p-balancedness is a very general property guaranteeing the existence of stable solutions, we have to mention that recognizing this property could be cumbersome and it is a two step process. First, a distribution has to be found, and then, the property of balance with respect to this distribution has to be carried out. An effective procedure to find a right distribution is not known yet. However, in the case it is already known that a matching problem is p-balanced, and a core stable matching $\mu*$ is available, a distribution I with respect to the matching problem is p-balanced is defined by

$$I(A) = \{t \in A : \mu^*(t) \geq_t A\},$$

as it follows easily from Iehlé (2007). Recognizing balancedness is in this sense easier since the first step is already solved.

A better understanding of p-balancedness would be fruitful to recognize new classes of matching problems having stable solutions. In this direction we have recently introduced the notion of partitioning hedonic game (Cesco (2010)), which is a hedonic version of the partitioning game developed in Keneko and Wooders (1982), and have used it to give another non-constructive proof of the existence of stable solutions in the marriage model of Gale and Shapley (1962) (see also Sotomayor (1996)).

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