Assignment models with capacity restriction, path to stability[1].

Abstract: In this paper we consider a special assignment model which involves two complementary types of agents (type I workers and type II workers) and an institution, which has preferences over the possible assignments. It can hire a set of pairs of complementary workers and has a quota q, which is the maximum number of candidates allowed to be hired. We study an algorithm whereby, starting from an arbitrary assignment of the model, this converges to a $q-$ stable assignment.

Keywords: matching ; quota ; stability ; restriction.

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1. Introduction

Bilateral assignment models are used to study problems of markets whose distinctive feature is that the agents involved are members of disjoint sets with different characteristics (e.g., employers and workers, hospitals and physicians, etc.). The nature of the problem studied here consists of assigning each member of a set to one agent of another set at most. Within these models the term *assignment* refers to the bilateral nature of the relation between the agents of these two sets.

The fundamental question of this assignment problem consists of the assignment of each worker from one side with a worker from the other side. Roth (1984, 1986, 1990 and 1991), Mongell and Roth (1991), Roth and Xing (1994), and Romero-Medina (1997) are examples of research work which investigates particular assignment problems like entry-level professional labor markets and student admissions at colleges.

The agents have preferences on the potential partners. Stability has been considered the main property to be satisfied by any sensible assignment. A assignment is called stable if all the agents are matched to an acceptable partner and there is no matched pair of workers that would prefer another partner to their current one.

The assignment model with capacity restriction consists of a model by means of which each worker from one side of the market is assigned one worker on the other side, such that the pairs of workers hired by the institution are q at most, that is to say, the institution will have to choose q pairs of workers at most according to their order of preference.

It should be noted that, even though this model involves two sets of workers and an institution, it is not equivalent to the trilateral allocation model introduced in 1986 by Alkan [1], in which there is no stability.

The property of stability in this model depends on the preferences expressed by the participants and the preference of the institution; for this reason, the property of $q-$stability is defined. Unfortunately, the set of $q-$stable assignments may be empty. However, under the restriction of the institution's responsive preferences, the existence of the set of $q-$stable assignments is guaranteed and it is possible to obtain its characterization. These results have been obtained with Femenia, Neme and Oviedo as joint authors.

We consider a assignment model with capacity constraint in which the organization has hired q pairs of workers at most and, for some reason (e.g, technological, budgetary, building-related, etc.), its hire quota varies and, as a consequence, a new model is obtained.

In this paper we show the relationship between subsets of stable assignments of these two models. We define the concept of maximal pair for an assignment and show a method to find all maximal pairs for an assignment established beforehand. In addition, we give an algorithm by which, starting from a model with a fixed quota, we can find all other stable assignments in the new model with capacity constraints, in which the given quota increases.

The paper is organized as follows. In Section 2 we present the notation and the definitions of the model. We introduce responsive preferences on assignments. We show that the set of $q-$stables is not empty and give its characterization. In Section we give an algorithm by which starting from an arbitrary assignment model, it converges to an assignment $q-$stable.

2. Preliminaries

2.1. The model

This model consists of two finite and disjoint sets of agents, i.e. the set of n type I workers and the set of m type II workers , denoted by $D=\{d_1,\ldots,d_n\}$ and $E=\{e_1,\ldots,e_m\}$, respectively. Each worker $d\in D$ has a strict preference relation[2] P_d over the set of agents $E\cup\{\phi\}$ and each worker $e\in E$ has a strict preference relation P_e over the set of agents $D\cup\{\phi\}$. Notice that only workers' strict preferences are being considered in his work. In some cases, similar results may be obtained if indifference is allowed. *Preference profiles* are $(n+m)$ - tuples of preference relations represented by $\mathbf{P}=(P_{d_1},\ldots,P_{d_n};P_{e_1},\ldots,P_{e_m})=(P_D,P_E)$. Given a preference profile $\mathbf{P}$, the standard assignment market is denoted by $M=(D,E,\mathbf{P})$ Given a preference relation P_d (P_e), the workers in the set preferred to the empty set by $d(e)$ are called *acceptables*. Similarly, given a preference relation P_e, the workers in the subsets of workers preferred to the empty set by e are called *acceptables*. The assignment problem consists of assignment type I workers with type II workers keeping the bilateral nature of their relationship as well as the possibility for both types of workers to remain unemployed. Formally:

Definition 1 A *assignment (matching)* μ is a mapping from set $D\cup E$ into set $D\cup E\cup\{\phi\}$, such that for all $d\in D$ and $e\in E$:

1. Either $\mu(d)\in E$ or $\mu(d)=\phi$.
2. Either $\mu(e)\in D$ or $\mu(e)=\phi$.
3. $\mu(d)=e$ if and only if $\mu(e)=d$.

Let $\mathcal{M}$ be the set of all possible assignments μ

Given a assignment market $M=(D,E,\mathbf{P})$, a assignment μ is blocked by a single agent $f\in D\cup E$ if $\phi P_f\,\mu(f)$. We say that a assignment is *individually rational* if it is not blocked by any single agent. A assignment μ is *blocked by a pair* of workers (d,e) if $dP_e\,\mu(e)$ and $eP_d\,\mu(d)$.

Definition 2 A assignment μ is *stable* if it is not blocked by any individual agent or by any pair of workers.

Given a assignment model $M=(D,E,\mathbf{P})$, $S(M)$ denotes the set of stable assignments. Notice that Gale and Shapley (1962) [4] proved that the set of stable assignments is non-empty, i.e., $S(M)\neq\phi$.

2.2. The assignment model with quota restriction

It consists of two disjoint sets of agents, the set of n type I workers and the set of m type II workers, denoted by $D=\{d_1,\ldots,d_n\}$, $E=\{e_1,\ldots,e_m\}$, respectively, and an institution denoted by U. Institution U has a binary relation R_U over the set of all possible assignments $\mathcal{M}$, the empty assignment included. Let P_U and I_U denote the strict and indifferent preference relations induced by R_U, respectively. The pair of workers will work for institution U ,which has a maximum number of positions – quota $q\leq\min\{n,m\}$ – to be filled; then, only the assignments whose cardinality is smaller or equal to q may be acceptable. The institution may choose some assignments of $\mathcal{M}$ according to its preference P_U and their quota restriction q. We denote $\mathcal{M}_q=\{\mu\in\mathcal{M}:\#\mu\leq q\}$.

[2] A preference is a binary, reflexive, antisymmetrical, transitive and complete relation.

This new assignment marker is denoted by $M_U^q = (M, R_U, q)$.

A assignment μ is *acceptable* for institution U according to their preferences if $\mu \in M_q$ and $\mu R_U \mu^{\phi}$, in which μ^{ϕ} is the assignment such that $\mu^{\phi}(x) = \phi$, for every $x \in D \cup E$.

Given M and a quota $q \leq \min\{n, m\}$, the institution can only accept assignments of M which are most preferred to the empty assignment according to its preference P_U, and its cardinal is not larger than the allowed number of positions $\# \mu \leq q$. A assignment is acceptable if the partner assigned is preferred to the empty set. Formally,

Definition 3 Given a model M_U^q, an assignment μ is $q-$ *individually rational* if $\# \mu \leq q$, $\mu P_U \mu^{\phi}$ and for every $f \in D \cup E$ such that $\mu(f) P_f \phi$ is verified.

Given an assignment $\mu \in$ M and a pair of workers $(d, e) \in D \times E$ we define $\mu_{(d,e)}$ as follows:

$$\mu_{(d,e)}(f) = \begin{cases} \mu(f) & if \quad f \notin \{d, e, \mu(d), \mu(e)\} \\ d & if \qquad f = e \\ \phi & \qquad otherwise. \end{cases}$$

Notice that, if $\mu(d) = e$, then $\mu_{(d,e)} = \mu$.

Note 1 The assignment $\mu_{(d,e)}$ may not be individually rational. Let us consider a assignment μ such that $\# \mu = q$ and let (d, e) such that, if $\mu(d) = \phi = \mu(e)$, then $\# \mu_{(d,e)} > q$ and $\mu_{(d,e)}$ is not $q-$individually rational.

Usually, in standard models, (d, e) is a blocking pair if these agents are not assigned to each other and if they each prefer one another to their current partners. Note that in our model, we may have a blocking pair (d, e) such that the assignment that the blocking pair is satisfying is not acceptable for institution U. Then, we will consider two types of blocking pairs for μ. One type is that which occurs when the assignment μ is blocked by a couple of agents in the institution, already assigned by the assignment, and the other is the type in which the assignment is blocked by a pair of agents, one of whom at least is single. In this case the assignment obtained, which satisfies the blocking pair, is preferred by the institution to the assignment μ. Formally:

Definition 4 A assignment μ is $q-$ *blocked* by a pair of workers (d, e) if $\mu(d) \neq e$ and

1. $e P_d \mu(d)$, $d P_e \mu(e)$, and
2. either (a) $\mu(d) \in E$ and $\mu(e) \in D$, or

   (b) $\mu_{(d,e)}$ is $q-$individually rational and $\mu_{(d,e)} R_U \mu$.

Definition 5 A assignment μ is $q-$ *stable* if it is $q-$individually rational and is not $q-$blocked by any pair of workers.

Given a assignment market $M_U^q = (M, R_U, q)$, $S(M_U^q)$ denotes the set of *q*-stable assignments. Notice that in Femenia [3] it was proved that, under the restriction of the institution's responsive preferences, the set of *q*-stable assignments is non-empty, i.e. $S(M_U^q) \neq \phi$. They also obtained a characterization of this set as: $S(M_U^q) = T_q(M) \cup T_{<q}(M)$.

Note: The definition of the institution's responsive preferences and of sets $T_q(M)$ and $T_{<q}(M)$ are given in detail in Appendix 1.

3. An algoritm that leads to a q-stable

Given an arbitrary assignment models with capacity restriction, we give an algorithm that allows us to find, with probability one, a q-stable assignment in this model.

3.1. Algorithm for finding a q-stable assignment

Considerer an assignment market $M=(D,E,\mathbf{P})$, in 1962, Gale and Shapley [4] proved that for any choice of agents, the set of stable assignment is not empty.

In 1976, Knuth [6] shows that given an assignment that is not stable, if we build a road of assignment satisfying blocking pairs, we can get into a cycle, i.e. the process of allowing the blocking pairs assign may not lead to a stable assignment. This left open the question: Is there at least one path to get from any assignment to a stable assignment for any choice of the agents?

Roth-Vande Vate [8] answered affirmatively the question posed by Knuth, when is not necessary that agents that were not assigned, when a block is satisfied should be reassigned to others. (Clearly such paths may not exist when different numbers of agents in sets D and E). Recall the theorem resolves this issue.

Let μ a assignment of $M=(D,E,\mathbf{P})$. Then exists a finite sequence of assignments $\mu_1,\mu_2,\dots,\mu_k$, such that $\mu=\mu_1$, $\mu_k\in S(M)$ and for each $i=1,\cdots,k-1$ exists (d_i,e_i) blocking pair for μ_i, such that μ_{i+1} obtained from μ_i satisfying blocking pair (d_i,e_i).

Idea of proof are given in detail in Appendix 2.

To view the application of the algorithm and Roth-Vande Vate, cite the example you gave in your work Knuth.

Example 1 Let $M=(D,E,\mathbf{P})$ with $D=\{d_1,d_2,d_3\}$ $E=\{e_1,e_2,e_3\}$, and let the following preferences be given by:

$$\begin{array}{ll} P_{d_1}=e_2,e_1,e_3 & P_{e_1}=d_1,d_3,d_2 \\ P_{d_2}=e_1,e_3,e_2 & P_{e_2}=d_3,d_1,d_2 \\ P_{d_3}\quad e_1,e_2,e_3 & P_{e_3}=d_1,d_3,d_2. \end{array}$$

Consider the assignment $\mu_1=\begin{pmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3\end{pmatrix}$, μ_1 is blocked by the pair (d_1,e_2) and (d_3,e_2). To apply this algorithm we choose the second pair, as $d_3P_{e_2}d_1$. Satisfying the pair (d_3,e_2) obtain $\mu_{1(d_3,e_2)}=\mu_2=\begin{pmatrix} d_1 & d_3 & d_2 & \phi \\ e_1 & e_2 & \phi & e_3\end{pmatrix}$, (d_2,e_3) blocks μ_2. Satisfying this pair found $\mu_{2(d_2,e_3)}=\mu_3=\begin{pmatrix} d_1 & d_2 & d_3 \\ e_1 & e_3 & e_2\end{pmatrix}$ is a stable.

Knut in the first step satisfies the pair (d_1, e_2) and after 8 blocks arrive again to μ_1, constructing a cycle of blocking pairs.

3.2. An algorithm to compute a q-stable in M_U^q.

Given $M_U^q = (M, R_U, q)$ a assignment models with capacity restriction, withn $M = (D, E, \mathbf{P})$ and R_U a preference responsive.

Suppose we have a $\mu \in M^{(t^*, s^*)}$, denote by $\gamma_{(t^*, s^*)}(\mu)$ the application of Roth y Vande-Vate [8] to this assignment in the model $M^{(t^*, s^*)}$.

To present the algoritm we will define an $q-$truncation for a pre-established assignment.

Definición 6 Given an assignment μ model M y $\succ_D$ the individual preference the institution. The assignment $\bar{\mu}$ is a *truncation* of μ with respect to d_t if $\bar{\mu}(d) = \begin{cases} \mu(d) & \text{si} \quad d \succeq_D d_t \\ \phi & \text{otherwise} \end{cases}$

Symmetrically defined in truncation of a assignment to agents E.

When # $\bar{\mu}$ =q we will say that $\bar{\mu}$ is an $q-$*truncation* of μ with respect to d_t.

In the following example we show how obtain one $q-$truncation of a given assignment.

Example 2 Let $\mu = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ e_2 & \phi & e_3 & e_1 \end{pmatrix}$ and individual preferences $d_1 \succ_D d_2 \succ_D d_3 \succ_D d_4$.

$\bar{\mu} = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 & \phi \\ e_2 & \phi & e_3 & \phi & e_1 \end{pmatrix}$ is a $2-$truncation of μ with respect to d_3.

The lemma, which we detail below, will be useful in the proof of a theorem later.

Lemma 1 Given $\mu \in S(M)$ such that $\#\mu > q$ and $\bar{\mu}$ the $q-$truncation of μ with respect to $d_{\bar{t}}$. If $\bar{\mu} \notin S(M_U^q)$, then $\#\gamma_{(\bar{t}, m)}(\bar{\mu}) \geq q$.

Proof See Appendix 2

Let μ_1 an arbitrary assignment in the model M_U^q, we need a procces whereby we find a new assignment, which belongs to $S(M_U^q)$. When the assignment is not stable in the model *M*, we apply the algorithm Roth-Vande Vate and find $\gamma_{(n,m)}(\mu_1) \in S(M)$, if its cardinality is less or equal to q, the process ends.

Otherwise, we choose $d_{\bar{t}_1}$ and make an $q-$truncation of $\gamma_{(n,m)}(\mu_1)$, which we call μ_2. If this truncation is stable, we find the assignment searched.

Otherwise, we apply the algorithm Roth-Vande Vate in the model $M^{(\bar{t}_1, m)}$. Note that $D^{\bar{t}_1} \subset D$. We repeat the previous prosses and get μ_3 assignment stable in $M^{(\bar{t}_2, m)}$ such that $\bar{t}_2 < \bar{t}_1$. Consecuently with the iterated application of this process we obtain a assignment $q-$stable.

Note that in each iteration procedure is applied to a subset of D strictly included in the previous one.

The following describes the steps to follow:

Let $\mu_1 \in \mathrm{M}_q$.

1. If $\mu_1 \in S(M_U^q)$ the process ends.

2. If $\mu_1 \notin S(M_U^q)$, then $\mu_1 \notin S(M)$ (because $\#\mu_1 \leq q$), then $\mu_2 = \gamma_{(n,m)}(\mu_1)$.

3. $i := 2$

4. Calculate $h = \#\mu_i$

4.1 Si $\#\mu_i \leq q$, the algoritm ends.

4.2 Si $\#\mu_i > q$, let d_t such that

a) $\mu(d_t) \neq \phi$.

b) $\#\{d \in D : d \succeq_D d_t \quad \text{y} \quad \mu(\mathrm{d}) \neq \phi\} = q$.

4.2.1 $\mu_{i+1}(d) = \begin{cases} \mu_i(d) & \text{if} \quad d \succeq_D d_t \\ \phi & \text{otherwise} \end{cases}$

4.2.2 If $\mu_{i+1} \in S(M^{(t,m)})$, then the algoritm ends.

4.2.3 If $\mu_{i+1} \notin S(M^{(t,m)})$. Let $\gamma_{(t,m)}(\mu_{i+1}) = \mu_{i+2} \in S(M^{(t,m)})$ (note that $\gamma_{(t,m)}(\mu_{i+1})$ is applying the algoritm Roth-Vande Vate in $M^{(t,m)}$).

5. Go to step 4.

Note: if we make the $q-$ truncation wich respect thr agents of E, we get another $q-$ stable.

Teorema 1 Let μ an assignment of M_q. Then exists a finite sequence of assignments $\mu_1, \mu_2, \cdots, \mu_k$, obtained by the above algorithm, such that $\mu_k \in S(M_U^q)$.

Proof See Appendix 2

The following example illustrate the operating of the last algorithm.

Example 3 Let $M_U^2 = (M, R_U, 2)$, $D = \{d_1, d_2, d_3, d_4, d_5\}$, $E = \{e_1, e_2, e_3, e_4\}$, with the following preferences:

$$\begin{array}{lll} P_{d_1} = e_4, e_3 & P_{d_4} = e_4, e_1, e_2 & P_{e_2} = d_2, d_1 \\ P_{d_2} = e_4, e_1, e_2 & P_{d_5} = e_4, e_1, e_2, e_3 & P_{e_3} = d_5, d_2, d_1 \\ P_{d_3} = e_1, e_4 & P_{e_1} = d_4, d_2, d_5 & P_{e_4} = d_3, d_1 \end{array}$$

and the following individual preferences $d_1 \succ_D d_2 \succ_D d_3 \succ_D d_4 \succ_D d_5$ y $e_1 \succ_E e_2 \succ_E e_3 \succ_E e_4$. Let $\mu = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & \phi & \phi \\ e_4 & \phi & e_3 & \phi & \phi & e_1 & e_2 \end{pmatrix}$.

$\mu = \mu_1 \in \mathrm{M}_2$, as $\mu_1 \notin S(M_U^2)$, it is not $2-$ individually rational for e_3 not is acceptable for d_3 and (d_3, e_1) is a blocking pair, we apply algorithm Roth-Vande Vate and obtain $\mu_2 = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 & d_5 \\ \phi & e_2 & e_4 & e_1 & e_3 \end{pmatrix}$.

As $h = \#\mu_2 = 4$ truncate the assignment μ_2, for this, let $d_t = d_3$ as $\mu_2(d_2) \neq \phi$, $\mu_2(d_3) \neq \phi$, $\#\{d_2, d_3\} = 2$ and we define

$$\mu_3 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi & \phi & d_4 & d_5 \\ \phi & e_2 & e_4 & e_1 & e_3 & \phi & \phi \end{pmatrix}.$$

$\mu_3 \notin S(M^{(3,4)})$ because the pair (d_1, e_3) blocks this assignment, we apply algorithm de Roth-Vande Vate and found $\mu_4 = \begin{pmatrix} d_1 & d_2 & d_3 & \phi & d_4 & d_5 \\ e_3 & e_1 & e_4 & e_2 & \phi & \phi \end{pmatrix}$.

We calculate $h = \#\mu_4 = 3$. As $\#\mu_4 > 2$ truncate the assignment, choosing $d_l = d_2$ and define $\mu_5 = \begin{pmatrix} d_1 & d_2 & \phi & \phi & d_3 & d_4 & d_5 \\ e_3 & e_1 & e_2 & e_4 & \phi & \phi & \phi \end{pmatrix}$.

As $\mu_5 \in S(M^{(2,4)})$, the algorithm ends with

$$\mu_5 \in T_2(M^{(2,4)}) \subseteq T_2(M) \subseteq T_{<2}(M) \cup T_2(M) = S(M_U^2).$$

4. Conclusions

In this research work, given a assignment model with restriction capacity, we obtain an algorithm by which, starting from any assignment of cardinal *q,* we find a stable assignment in the model considered. We define the concept of $q-$truncation and describe an algorithm, by which application we can get the assignments of the set of stables model in which we work.

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APPENDIX I - The restriction of M

From now on, we will denote $F \in \{D,E\}$ and $F^c \in \{D,E\}$ such that $\{F,F^c\} = \{D,E\}$, and $f \in F$ will denote a generic worker. Given $F' \subseteq F$, we **denote** the restriction of P_F to F' by $P_{F'}$. Given $M = (F, F^C, \mathbf{P})$, we denote the restriction of M to F' by $M_{F'} = (F', F^C, P_{F'}, P_{F^C})$. For the sake of simplicity we denote $M_{F'} = (F', F^C, \mathbf{P})$ where we have to understand that $\mathbf{P} = (P_{F'}, P_{F^C})$.

Lemma A1 *(Femenia, Marí, Oviedo and Neme 2011)* Given $M = (D, E, \mathbf{P})$ and $F' \subseteq F$, let μ and μ' be the stable assignments for M and $M_{F'}$ respectively. Then $\#\mu' \leq \#\mu \leq \#\mu' + \#(F \setminus F')$.

The institution's responsive preference

Given a assignment market M_U and a quota $q \leq \min\{n,m\}$, we denote a $S(M_U^q)$ the set of all $q-$stable assignments. We will assume that the institution has an individual preference $\succ_D$ over the set $D \cup \phi$ and an individual preference $\succ_E$ over the set $E \cup \phi$ and its preference over assignments are directly connected with its preferences over workers. An institution's preference is called responsive to its individual preferences if, for any assignment that differs in only one worker, the institution prefers the assignment that has the most preferable worker according to the individual preferences.

In order to formalize the institution's responsive preference, we introduce the notations that follow.

For every assignment μ and $f \in D \cup E$, consider $B_\mu = \{(d,e) \in D \times E : \mu(d) = e\}$ and we define $\mu^{(d,e)}(f) = \begin{cases} \phi & if \quad f \notin \{d,e\} \\ d & if \quad f = e. \end{cases}$ Notice that $\mu^{(d,e)} = \mu^{\phi}_{(d,e)}$.

Definition A1 A preference relation R_U is *a responsive extension* of preferences $\succ_D$ and $\succ_E$ over $D \cup \{\phi\}$ and $E \cup \{\phi\}$ respectively, such that it satisfies the following conditions:

i) $\mu^{(d,e)} P_U \mu^{\phi}$ if and only if $d \succ_D \phi$ and $e \succ_E \phi$.

ii) $\mu P_U \mu^{\phi}$ if and only if $\mu^{(d,e)} P_U \mu^{\phi}$ for every $(d,e) \in B_\mu$.

iii) $\mu^{(d,e)} P_U \mu^{(d,e')}$ if and only if $e \succ_E e'$.

iv) $\mu^{(d,e)} P_U \mu^{(d',e)}$ if and only if $d \succ_D d'$.

v) For every $\mu, \mu' \in$ M such that $\#\mu = \#\mu'$ and $B_\mu = B_{\mu'} \setminus \{(d',e')\} \cup \{(d,e)\}$: $\mu P_U \mu'$ if and only if $\mu^{(d,e)} P_U \mu^{(d',e')}$.

vi) For every $\mu, \mu' \in$ M such that $B_{\mu'} \subset B_\mu$ and $\mu P_U \mu^{\phi}$, then $\mu P_U \mu'$.

vii) For every $\mu, \mu' \in$ M such that $\mu(E) = \mu'(E)$ and $\mu(D) = \mu'(D)$, then $\mu \, I_U \mu'$.

We consider a preference R_U to be responsive if there are two individual preferences $\succ_D$ and $\succ_E$ over $D \cup \phi$ and $E \cup \phi$ respectively, such that R_U is a responsive extension.

Remark A1 Given two preferences $\succ_D$ and $\succ_E$, over $D \cup \phi$ and $E \cup \phi$ respectively, we can construct a responsive preference relation R_U over the set of all assignments M ; moreover, this extension is not unique.

The setes $T_q(M)$ and $T_{<q}(M)$

Now we will consider the model M_U^q, where R_U is a responsive preference. Without loss of generality and in order to avoid the addition of notational complexity to the model M_U^q, we assume that all the agents of sets D and E are acceptable for the institution, i.e. for every $d \in D$ and $e \in E$, we have that $d \succ_D \phi$ and $e \succ_E \phi$.

For every $t \in \mathrm{N}$, we can define the following subset $F^t \subseteq F$ such that $\# F^t = t$, and for every $f \in F^t$ and $f' \notin F^t$ we have that $f \succ_F f'$. Note that $F^1 \subset F^2 \subset \ldots \subset F^l = F$ where $\# F = l$.

Given sets $\mathbf{d} = \{1,2,\ldots,\# D\}$ and $\mathbf{e} = \{1,2,\ldots,\# E\}$; for every $(t^*,s^*) \in \mathbf{d} \times \mathbf{e}$, we denote $M^{(t^*,s^*)}$, the restriction of M to D^{t^*} and E^{s^*}, i.e. $M^{(t^*,s^*)} = \left(D^{t^*}, E^{s^*}, \mathbf{P}\right)$.

Given $(t^*,s^*) \in \mathbf{d} \times \mathbf{e}$, q, and the following sets of assignments:

$$T_q(M^{(t^*,s^*)}) = \begin{cases} S(M^{(t^*,s^*)}) & if \quad \# \mu = q \text{ for every } \mu \in S(M^{(t^*,s^*)}) \\ \phi & \text{otherwise} \end{cases}$$

And $T_q(M) = \{\mu : \exists \quad (t^*,s^*) \text{ such that } \mu \in T_q(M^{(t^*,s^*)})\}$.

Proposition A4 Given $M_U = (M, R_U)$, $(t^*,s^*) \in \mathbf{d} \times \mathbf{e}$, there exists $K \subseteq \mathbf{d} \times \mathbf{e}$, such that $T_q(M) = \overset{?}{\bigcup\limits_{(t^*,s^*)? K}} T_q(M^{(t^*,s^*)})$

Given $(t^*,s^*) \in \mathbf{d} \times \mathbf{e}$, q and the following sets of stable assignments:

$$T_{<q}(M^{(t^*,s^*)}) = \left\{ \mu \in S(M^{(t^*,s^*)}) : \# \mu < q, \text{either } \phi P_e d \text{ or } \phi P_d e \text{ for every } d \in D \setminus \mu(E^{s^*}) \text{ and } e \in E \setminus \mu(D^{t^*}) \right\}$$

and $T_{<q}(M) = \{\mu : \exists \quad (t^*,s^*) \text{ such that } \mu \in T_{<q}(M^{(t^*,s^*)})\}$.

Proposition A5 Given $M_U = (M, R_U)$, $(t^*,s^*) \in \mathbf{d} \times \mathbf{e}$, there exists $\hat{K} \subseteq \mathbf{d} \times \mathbf{e}$, such that $T_{<q}(M) = \dot{\bigcup\limits_{(t^*,s^*) \in \hat{K}}} T_{<q}(M^{(t^*,s^*)})$.

Remark A2 The sets K and $\hat{K}$ on the previous propositions are given by:

$K = \{(t^*,s^*) \in N : T_q(M^{(t^*,s^*)}) \cap T_q(M^{(t',s')}) = \phi$ forall $(t',s') < (t^*,s^*)\}$ and

$\hat{K} = \{(t^*,s^*) \in N : T_q(M^{(t^*,s^*)}) \cap T_q(M^{(t',s')}) = \phi$ forall $(t',s') < (t^*,s^*)\}$,

Now, we are going to present the following results which state that the set of $q-$ stable assignment are non-empty.

Teorem A1 If $M_U^q, = (M, R_U, q)$ is a assignment market, then $S(M_U^q) \neq \phi$.

The following theorem is a complete characterization of the $q-$stable sets $S(M_U^q)$.

Teorem A2 If $M_U^q, = (M, R_U, q)$ is a assignment market, then $S(M_U^q) = T_q(M) \cup T_{<q}(M)$.

APPENDIX 2 - Theorem B1(Roth-Vande Vate).

Let μ an assignment of $M=(D,E,\mathbf{P})$. Then exists a finite sequence of assignments $\mu_1,\mu_2,\ldots,\mu_k$, such that $\mu=\mu_1$, $\mu_k\in S(M)$ and for each $i=1,\cdots,k-1$ exists (d_i,e_i) blocking pair for μ_i, such that μ_{i+1} obtained from μ satisfying blocking pair (d_i,e_i).

Idea of proof : Is μ_1 an arbitrary assignment individually rational (if μ_1 not individually rational, we can start the sequence with a chain of blocking pairs for individual locks to achieve individually rational assignment). If μ_1 stable the theorem holds trivially. Suppose there is exists a pair (d_1,e_1) that bolcks μ_1, is $\mu_2=\mu_{1(d_1,e_1)}$ and define the set $A(1)=\{d_1,e_1\}$. Note that if (d_2,e_2) any pair blocker μ_2 then $\{d_2,e_2\}\subset A$.(1)

The proof is constructed in each step a sequence of assignments associated with an increasing sequence of sets $A(k)$ that do not contain agents form pairs blocking, until it has obtained a stable assignment.

Inductively, suppose we have a set $A(k)$ such that exists blocking pairs for μ_{k+1} contents in $A(k)$, and such that μ_{k+1} not assign an agent $A(k)$ to an agent that is not in $A(k)$. Then if μ_{k+1} is not stable, exists a pair blocker (d',e') such that at most d' or $e'\in A(k)$. We consider three cases:

Case 1 exists a blocking pair (d_{k+1}, e_{k+1}) such that $d_{k+1}\in A(k)$. We choose (d_{k+1}, e_{k+1}) such that between all pairs blockers (d, e_{k+1}), d_{k+1} is the agent of $A(k)$ most preferred for e_{k+1}. Let μ_{k+2} the following assignment of the sequence, with $\mu_{k+2}=\mu_{k+1(d_{k+1}, e_{k+1})}$. We define $A(k+1)=A(k)\cup\{e_{k+1}\}$.

i) If d_{k+1} is such that $\mu_{k+1}(d_{k+1})=\phi$, then $A(k+1)$ is a set such that no blocking pair for μ_{k+2} is contained in it.

ii) On the other hand, may be a blocking pair (d_{k+2}, e_{k+2}) for μ_{k+2}, with $e_{k+2}=\mu_{k+1}(d_{k+1})$ and $d_{k+2}\in A(k+1)$.

In this case we choose the pair blocker with the property that all blocking pairs (d, e_{k+2}), d_{k+2} is the most preferred agent e_{k+2} in $A(k+1)$. The following assignment in sequence μ_{k+3}, is formed by satisfying the blocking pair and the process continues until we reach an assignment μ_r with $r>k$, such that any blocking pair of μ_r is contained in the set $A(r)=A(k+1)$. This should happen eventually, because any d receive a partner worse than the one that had before. Then any blocking pair appears twice in the sequence.

The set $A(r)$ is the set that we require, it contains strictly to $A(k)$ and does not contain blocking pairs for μ_r

Case 2 exists a blocking pair (d_{k+1}, e_{k+1}) such that $e_{k+1}\in A(k)$. Proceed as in the previous case, invirting the roles of the sets D and E.

Caso 3 each blocking pair (d_{k+1}, e_{k+1}) is disjoint with $A(k)$. We define the set $A(k+1)=A(k)\cup\{m_{k+1}, w_{k+1}\}$ with $A(k)\subseteq A(k+1)$. The procces must end in a finite number of steps (because $A(k)$ can grow indefinitely until a stable assignment, but can not be larger than $D\cup E$. So we have reached an assignment. This completes the proof.

Lemma B1 Given $\mu\in S(M)$ such that $\#\mu>q$ and $\bar{\mu}$ the $q-$truncation of μ with respect to $d_{\bar{i}}$. If $\bar{\mu}\notin S(M_U^q)$, then $\#\gamma_{(\bar{i},m)}(\bar{\mu})\geq q$.

Proof Let $\mu_1 = \overline{\mu}$ the $q-$truncation of μ with respec to $d_{\bar{t}}$. As $\#\overline{\mu} = q$ and $\overline{\mu} \notin S(M^{(\bar{t},m)})$, as $D^{t} \subset D$, $\overline{\mu}$ is individually rational in this model, then exists a pair $(d,e) \in D^{t} \times E$ blocking $\overline{\mu}$.

We will apply the algorithm Roth- Vande Vate to satisfy the blocking pairs that we find. It should be noted that in this algorithm to find $\overline{\mu}_{(d,e)}$, the agent d we choose is the best for the preferences of e that blocks with him. In the pairs we are going to we will satisfy, it can happen:

Case 1 $\overline{\mu}(e) = \phi$. Then by definition 6 $\overline{\mu}(e) = \mu(e)$ and as for all $d \succeq_D d_{\bar{t}}$ the assignments μ and $\overline{\mu}$ coincide, then the pair (d,e) blocks μ, which is a contradiction.

Case 2 $\overline{\mu}(e) = \phi$. **Caso 2.1** $\overline{\mu}(d) \neq \phi$. As $\overline{\mu}(e) = \overline{\mu}(d) = \phi$, satisfying the blocking pair we are $\mu_2 = \overline{\mu}_{(d,e)}$ and $\#\overline{\mu}_{(d,e)} = q+1$. If μ_2 stable, then $\mu_2 = \gamma_{(\bar{t},m)}(\overline{\mu})$. Note that $\#\gamma_{(\bar{t},m)}(\overline{\mu}) \geq q$. If μ_2 not stable seek another pair blocker and the process continues.

Case 2.2 $\overline{\mu}(d) \neq \phi$. As $\overline{\mu}(e) = \phi$ y $\overline{\mu}(d) \neq \phi$, satisfying the blocking pair we are $\mu_2 = \overline{\mu}_{(d,e)}$ and $\#\overline{\mu}_{(d,e)} = q$. If μ_2 stable, $\mu_2 = \gamma_{(\bar{t},m)}(\overline{\mu})$ with $\#\gamma_{(\bar{t},m)}(\overline{\mu}) = q$.

If μ_2 not stable, is (d', e') a blocking pair for μ_3 in $M^{(\bar{t},m)}$, then $\mu_2(e') = \phi$. In efecct, if $\mu_2(e') \neq \phi$ it can happen:

- If $e' = e$, then $d' P_{e'} \, d \, P_e \, \overline{\mu}(e)$ and as $e = e' P_{d'} \, \overline{\mu}(d')$, then (d', e) is blocking pair for $\overline{\mu}$ and this contradicts that in the choice of the first blocking pair, d is the most prefered to e of those with him are blocking pair.
- If $e' \neq e$, because μ_2 assigned as $\overline{\mu}$ agents except $\{d, e, \overline{\mu}(d)\}$ and $\mu_2(\overline{\mu}(d)) = \phi$, then $\mu_2(e') = \overline{\mu}(e')$, $\mu_2(d') = \overline{\mu}(d')$ and consequently (d', e') blocker $\overline{\mu}$ which is a contradiction.

Now, (d', e') is such that $\mu_2(e') = \phi$, when analyzing the cases $\mu_2(d') = \phi$ and $\mu_2(d') \neq \phi$, by satisfying the blocking we find assignments the cardinal $q+1$ and q respectively.

$\mu_3 = \mu_{2(d',e')}$ may be or not stable, following a similar process of satisfying blocking pairs get $\#\gamma_{(\bar{t},m)}(\overline{\mu}) \geq q$.

Teorema B1 Let μ an assignment of M_q. Then exists a finite sequence of assignments $\mu_1, \mu_2, \cdots, \mu_k$, obtained by the above algorithm, such that $\mu_k \in S(M_U^q)$.

Proof Let $\mu = \mu_1$. If $\mu_1 \in S(M_U^q)$, the theorem holds trivially. As $\#\mu_1 = q$, if $\mu_1 \notin S(M_U^q)$, we apply algorithm Roth-Vande Vate in M and found $\gamma_{(n,m)}(\mu_1) = \mu_2 \in S(M)$.

We consider the followings cases:

Case 1 $\#\mu_2 \leq q$. As $\mu_2 \in S(M)$ and $\#\mu_2 \leq q$, then $\mu_2 \in S(M_U^q)$, which complete the proof for this case.

Case 2 $\#\mu_2 > q$. As $\#\mu_2 > q$, perform a q-truncación of μ_2, e.i., is d_t such that $\mu(d_t) \neq \phi$ and $\#\{d \in D : d \succeq_D d_t \quad \text{y} \quad \mu(\text{d}) \neq \phi\} = q$. We define $\mu_3(d) = \begin{cases} \mu_2(d) & \text{si} \quad d \succeq_D d_t \\ \phi & \text{otherwise.} \end{cases}$ We consider the reduced model $M^{(t,m)}$ and if $\mu_3 \in S(M^{(t,m)})$, as $\#\mu_3 = q$, then $\mu_3 \in S(M_U^q)$. If $\mu_3 \notin S(M^{(t,m)})$, we apply algorithm Roth-Vande Vate and found $\gamma_{(t,m)}(\mu_3) = \mu_4 \in S(M^{(t,m)})$.

As $\mu_2 \in S(M)$ such that $\#\mu_2 > q$ and μ_3 q-truncation of μ_2 with $\mu_3 \notin S(M^{(t,m)})$, by lemma 1, $\#\mu_4 = \#\gamma_{(t,m)}(\mu_3) \geq q$. We consider the followings subcases:

Case 2.1 $\#\mu_4 = q$. As $\mu_4 \in S(M^{(t,m)})$ and $\#\mu_4 = q$, then $\mu_4 \in S(M_U^q)$.

Case 2.2 $\#\mu_4 > q$. As $\#\mu_4 > q$ we return to perform a q-truncation of the assignment, as previously described, for this, of the μ_4 select d_l and define μ_5. Let $L = \{d \in D^t : d \succeq_D d_l \quad \text{y} \quad \mu_4(d) \neq \phi\}$ y $\#L = q$. Since $\#\mu_4 > q$, the agent d_l found is such that $d_l \succ_D d_t$ and consequently $D^l \subset D^t$. (2)

μ_5 is a assignment of the model $M^{(l,m)}$, if $\mu_5 \in S(M^{(l,m)})$, as $\#\mu_5 = q$, then $\mu_5 \in S(M_U^q)$. If $\mu_5 \notin S(M^{(l,m)})$, is $\mu_6 = \gamma_{(l,m)}(\mu_5)$. The lemma B1 guarantees that $\#\mu_6 \geq q$. If $\#\mu_6 = q$ it following as case 2.1. If $\#\mu_6 > q$ it following as case 2.2.

Note that in each iteration we obtain a model reduced such that the set of agents D considered is strictly included in the previous set (2), this guarantees that at most in $n - q$ steps we obtain the desired stable assignment.