The core in the matching model with quota restriction*

Abstract: In this paper we study the core in the matching model with quota restriction, where an institution has to hire a set of pairs of complementary workers, and has a quota that is the maximum number of candidate pair positions to be filled. We define as natural both the concept of blocking by coalition and the concept of core. Under the restriction of the institution's responsive preferences the existence of core is guaranteed and its characterization is obtained.

Keywords: Matching; Stability; Core.

JEL classification: C78; C71; D79.

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1 Introduction

One-to-one matching models have been useful for studying assignment problems with the distinctive feature that agents can be divided from the very beginning into two disjoint subsets of complementary workers: the set of workers of type I and the set of workers of type II. The nature of the assignment problem consists of matching each agent (workers of type I) with one agent from the other side of the market (workers of type II).

The fundamental question of this assignment problem consists of matching each worker with a one worker from the other side. Roth (1984, 1986, 1990, and 1991), Mongell and Roth (1991), Roth and Xing (1994), and Romero-Medina (1997) are examples of research work which investigate particular matching problems like entry-level professional labor markets and student admission at colleges.

The agents have preferences on the potential partners. Stability has been considered the main property to be satisfied with any sensible matching. A matching is called stable if all the agents are matched to an acceptable partner and there is no matched pair of workers that would prefer another partner to their current one.

When the models of bilateral assignment are studied, the theory of *cooperative games* is relevant since the solution in these models has typical features of solution in such games. In cooperative games the problem is centered on the analysis of the possible solutions and its stability.

In these games some players (agents) may reach binding agreements, for in them the results that each of the coalitions of players that can be formed are studied. *Core* is a solution of a cooperative game that is not blocked by any coalition of agents and will be specified later.The difference between the definition of core and that of a group of stable matchings for a bilateral game lies in the fact**s** that the core is defined via a coalition of agents and that all coalitions play a role, while the set of stable matchings is defined only with respect to a certain type of coalition, that is to say, a result is in the core if it is not blocked by an agent or by any coalition of agents, the number of agents that intervene in the coalition not being considered. Some results referring to the non-empty core in a family of generalizations in the assignment market – which includes the market of marriage – are found in Quinzii (1984), Curiel and Tijs (1985), among others. Gale and Shapley (1962) show that the set of stable matchings and the core coincide.

A variant of the model of bilateral assignment is *the assignment model with quota restriction,* presented by Femenia, Mar, Neme and Oviedo (2008), where an institution has to hire a set of pairs of complementary workers, and has a quota (quota q) which is the maximum number of pair positions to be filled. This institution has a preference on this potential set of pairs. The property of stability in this model depends on the preferences expressed by the participants and on the preferences of the institution; that is why the property of q-stability is defined. Under the restriction of the institution's *responsive preference*, the existence of the set of q-stable matchings is guaranteed and its characterization is obtained.

In this research work the core in the assignment model with quota retriction is studied. For this aim the institution's role in the concept of blocking by coalition is reconsidered and the concepts of *q-block by coalition* and *q-core* are defined. Under the restriction of the institution's responsive preferences, the existence of the *q-core* is proved and, even though in this model the set of q-stable matchings and the q-core do not coincide, a complete characterization of the *q-core* is obtained.

The paper is organized as follows. Section 2 presents a brief revision of the theoretical concepts of the matching model and a result is stated that links the set of stable matchings to the core. The most important definitions of the assignment model with quota restriction and the results that guarantee the existence of the set of matching q-stable and its characterization are stated as well. In Section 3 we define the concepts of q-block by coalition and q-core. In Section 4 we show a result which links the set *of* q-stable matchings to the q-core as well as the characterization of this set. In Section 5, under the restriction of the institution's responsive preferences, the existence of the q-core is guaranteed. Section 6 presents this paper's conclusions.

2 Preliminaries

2.1 The matching model

It consists of two disjoint sets of agents, i.e., the set of n workers of type I, and the set of m workers of type II, denoted by $D=\{d_1,...,d_n\}$, $E=\{e_1,...,e_m\}$, respectively.

Each worker $d \in D$ has a strict preference relation P_d over $E \cup \{\varnothing\}$ and each worker $e \in E$ has a strict preference relation P_e over $D \cup \{\varnothing\}$.

Notice that only strict preferences are being considered. Similar results may be obtained if indifference is allowed.

Preference profiles are $(n+m)$-tuples of preference relation, represented by $\mathbf{P} = \left(P_{d_1},...,P_{d_n};P_{e_1},...,P_{e_m}\right) = (P_D, P_E)$.

Given a preference profile $\mathbf{P}$, the standard matching market is denoted by $M - (D, E, \mathbf{P})$.

Given a preference relation P_d, the subset of workers preferred to the empty set by d are called *acceptable*. Similarly, given a preference relation P_e, the subsets of workers preferred to the empty set by e are called *acceptable*. The assignment problem consists of matching workers of type I with workers of type II, the nature of the relationship being kept and there being the possibility for both types of workers to remain unmatched. Formally:

Definition 1 *A **matching** μ is a mapping from the set $D \cup E$ into the set $D \cup E \cup \{\varnothing\}$ such that for all $d \in D$ and $e \in E$:*

1. Either $\mu(d) \in E$ or $\mu(d) = E\,\varnothing$.
2. Either $\mu(e) \in D$ or $\mu(e) = E\,\varnothing$.
3. $\mu(d) = e$ if and only if $\mu(e) = d$.

Let **M** be the set of all possible matching μ.

Given a matching market $M = (D, E, \mathbf{P})$ a matching μ is blocked by a single agent $f \in D \cup E$ if $\varnothing P_f \mu(f)$. We say that a matching is *individually rational* if it is not blocked by any single agent. A matching μ is *blocked by a pair* of workers (d, e) if $dP_e \mu(e)$ and $P_d \mu(d)$.

Definition 2 *A matching μ is **stable** if it is not blocked by any individual agent or by any pair of workers.*

Given a matching market $M = (D, E, \mathbf{P})$, $S(M)$ denotes the set of stable matchings. Notice that Gale and Shapley (1962) [2] have proved that the set of stable matchings is non-empty, $S(M) \neq \varnothing$.

Given a matching market on an $M = (D, E, \mathbf{P})$, a matching μ is *blocked by a coalition* $S \subseteq E \cup D$, if there exists $\mu' \in$ **M**, such that:

1. $\mu'(S) \subseteq S$, and
2. $\mu'(x) P_x \mu(x)$ for all $x \in S$.

Definition 3 *The **core** of $M = (D, E, \mathbf{P})$ is the set of matchings of **M** if it is not blocked by any coalition.*

Given a matching market, $M = (D, E, \mathbf{P})$, $C(M)$ denotes the core of M.

One of the central results, proved by Roth and Sotomayor en 1990 [7], which links the core with the set of stable matchings, is that they coincide: $C(M) = S(M)$.

2.2 The matching model with quota restriction

It consists of two disjoint sets of agents, the set of n workers of type I and the set of m workers of type II, denoted by $D=\{d_1,...,d_n\}$, $E=\{e_1,...,e_n\}$, respectively, and an institution denote by U. Institution U has a meflexive, transitive, antisymmetric and complete binary relation R_U over the set of all possible matchings $\mathbf{M}$, the empty matching included. As usual, let P_U and I_U denote the strict and indifferent preference relations induced by R_U, respectively. The pair of workers will work for institution U and this has a maximum number of positions, quota $q \leq \min\{n,m\}$, to be filled; then, only the matchings whose cardinality is smaller or equal to q may be acceptable. The institution may choose some matchings of $\mathbf{M}$ according to their preference P_U and their quota restriction q. We denote $\mathbf{M}_q = \{\mu \in \mathbf{M}: \# \mu \leq q\}$.

This new matching marker is denoted $M_U^q = (M; R_U, q)$.

Notice that if $q \geq \min\{m,n\}$ the set of all the matchings may be acceptable, i.e., $\mathbf{M} = \mathbf{M}_q$.

A matching μ is *acceptable* for institution U according to their preferences if $\mu \in \mathbf{M}_q$ and $\mu R_U \mu^{\varnothing}$, where $\mu^{\varnothing}$ is the matching such that $\mu^{\varnothing}(x)=\varnothing$, for every $x \in D \cup E$.

Given M_U^q, $q \leq \min\{n,m\}$, a matching μ is $q-$ *individually rational* if $\# \mu \leq q$ $\mu P_U \varnothing$ and $\mu(f) P_f \varnothing$ for every worker, $f \in D \cup E$ such that $\mu(f) \neq \varnothing$. A matching μ is $q-$ *blocked* by pair of workers (d,e) if

1. $eP_d\mu(d)$, $dP_d\mu(e)$, and
2. either
   (a) $\mu(d) \in E$ and $\mu(e) \in D$, or
   (b) $\mu_{(d,e)}$ is $q-$ individually rational and $\mu_{(d,e)} R_U \mu$,

to $\mu_{(d,e)}$ defined by: $\mu^{(d,e)}(f) = \begin{cases} \mu(f) & \textit{if} \quad f \notin \{d, e, \mu(d), \mu(e)\} \\ d & \textit{if} \quad f = e \\ e & \textit{if} \quad f = d \\ \phi & \quad \textit{otherwise}. \end{cases}$

Notice that, if $\mu(d) = e$, then $\mu_{(d,e)} = \mu$

Definition 4 *A matching* μ *is* $q-$ ***stable*** *if it is* $q-$ *individually rational and is not* $q-$ *blocked by any pair of workers.*

Given a matching market $M_U^q = (M; R_U, q)$, $S(M_U^q)$ denotes the set of q-stable matchings. Notice that Femenia et al., (2008) [1] have proved that, under the restriction of the institution's responsive preferences, the set of q-stable matchings is non-empty, $S(M_U^q) \neq \varnothing$. They also obtained a characterization of such a set as: $S(M_U^q) = T_q(M) \cup T_{<q}(M)$.

Note: The definition of the institution's responsive preferences and the sets $T_q(M)$ and $T_{<q}(M)$ are made explicit in detail in the Appendix.

3 The *q*-core

Our objetive, in this section, is to extend the concept of blocking for coalition to the matching model with restriction of the capacity M_U^q. Unlike what happens in model M, in model M_U^q the institution U has preferences over the set of matchings $\mathbf{M}$; that is why it is necessary to reconsider the institution's role in the concept of blocking by coalition. We will consider that, when the coalition that blocks a matching is formed by agents already contracted by the institution, that is to say, not single agents, the institution does not raise any objection in its formation because it considers the blocking to be internal. On the other hand, if external agents belong to the coalition, that is to say, single agents in the original matching, the institution imposes the new matching to be preferred to the original one. Formally:

Definition 5 *A matching* μ *is* q *-blocked by a coalition* $S \subseteq E \cup D$ *, if exists* $\mu' \in \mathbf{M}$ *such that:*

1. $\mu'(S) \subseteq S$
2. $\mu'(x) P_x \mu(x)$ for all $x \in S$.
3. $\mu'(x) = \mu(x)$ for all $x \notin S \cup \mu(S)$.
4. If $x \in S$ such that $\mu(x) = \varnothing$, then $\mu' R_U \mu$.

In model M the core was defined as the set of matchings which are not blocked by any coalition. We extend the concept of core to the model M_U^q in the following way:

Definition 6 *Given a matching market whith quota restriction* M_U^q*, the* q ***-core*** *of* M_U^q *is the set of matchings of* $\mathbf{M}_q$ which are not q *-blocked by any coalition.*

We denote $C(M_U^q)$ the q -core of M_U^q .

Proposition 1 *Let* $M_U^q = (M, R_U, q)$ *be a matching market with quota restriction; if* $\mu \in C(M_U^q)$*, then* μ *es* q *-individually rational.*

Proof. Assume that μ is not q -individually rational, then there exists $f \in D \cup E$ such that $\varnothing P_f \mu(f)$. We consider the coalition $S = \{f\}$ and the matching μ, such that

$$\mu'(x) = \begin{cases} \mu(x) & if \quad x \notin \{f, \mu(f)\} \\ \phi & if \quad x \in \{f, \mu(f)\} \end{cases}$$

They satisfy the conditions of q -blocked by coalition, contradicting the fact that $\mu \in C(M_U^q)$.

The following proposition establishes that $T_q(M)$ is a subset of q -core.

Proposition 2 *If* $M_U^q = (M, R_U, q)$ *is a matching market with quota restriction, then* $T_q(M) \subseteq C(M_U^q)$.

The proof to this proposition is developed in the Appendix as Proposition A7 .

The following example shows that $T_{<q}(M)$ not always is a subset of $C(M_U^q)$.

Example 1 *Let* $M = (D, E, \mathbf{P})$ *be the matching market such that* $D=\{d_1,d_2\}$ *and* $E=\{e_1,e_2\}$ *are the two sets of workers with the preference profile* $(P_{d_1},..., P_{e_3})$*, where:*

$P_{d_1} = e_2, e_1$ $\quad$ $P_{e_1} = d_2, d_1$

$P_{d_2} = e_1$ $\quad$ $P_{e_2} = d_1$.

and $\succ_D$ and $\succ_E$ the following preferences over D and E:
$e_1 \succ_E e_2$ y $d_1 \succ_D d_2$.

Set $T_{<2}(M)$ consists of the following matching: $\mu = \begin{pmatrix} d_1 & d_2 & \varnothing \\ e_1 & \varnothing & e_2 \end{pmatrix}$.

We consider the coalition $S=\{d_1,d_2,e_1,e_2\}$.

By item vii) of the responsive extension $\mu' I_U \mu''$, where $\mu'' = \begin{pmatrix} d_1 & d_2 \\ e_1 & e_2 \end{pmatrix}$.

As $B_\mu \subset B_{\mu''}$, by item vi) of the responsive extension, $\mu'' P_U \mu$. Then $\mu' P_U \mu$ and $\mu \notin C(M_U^2)$.

We define the following subset of $T_{<q}(M)$ as follows:

$R = \{\mu \in T_{<q}(M)$: there exists d_1, d_2, e_1, e_2, such as $\mu(d_1) = e_1$ and $\mu(d_2) = \varnothing$, $\mu(e_2) = \varnothing, d_2 P_{e_1} d_1, e_2 P_{d_1} e_1, d_1 P_{e_2} \varnothing$ and $e_1 P_{d_2} \varnothing\}$.

We show that every matching that is in R is not in the q -core.

Proposition 3 *Let* $M_U^q = (M, R_U, q)$ *be a matching market and* $\mu \in R$. *Then* $\mu \notin C(M_U^q)$.

Proof. Let $\mu \in R$, then there exists d_1, d_2, e_1, e_2 such that $\mu(d_1)=e_1$, $\mu(d_2)=\varnothing$, $\mu(e_2) = \varnothing, d_2 P_{e_1} d_1, e_2 P_{d_1} e_1, d_1 P_{e_2} \varnothing$ and $e_1 P_{d_2} \varnothing$.

We consider the coalition $S=\{d_1,d_2,e_1,e_2\}$ and the matching

$$\mu'(x) = \begin{cases} \mu(x) & if \quad x \notin S \\ d_1 & if \quad x = e_2 \\ d_2 & if \quad x = e_1. \end{cases}$$

We have that $\mu'(S) \subseteq S$ and $\mu(x) = \mu'(x)$, for every $x \notin S \cup \mu(S)$. This the conditions 1) and 2) of q -blocked for coalition are satisfied. Also, conditions 3) of q -blocked for coalition are satisfied, since $\mu(d_1)=e_1$, $\mu(d_2)=\varnothing$, $\mu(e_2) = \varnothing, d_2 P_{e_1} d_1, e_2 P_{d_1} e_1, d_1 P_{e_2} \varnothing$ and $e_1 P_{d_2} \varnothing$.

To demonstrate condition 4), we consider the matching

$$\mu''(x) = \begin{cases} \mu'(x) & if \quad x \notin S \\ d_1 & if \quad x = e_1 \\ d_2 & if \quad x = e_2. \end{cases}$$

By tems vii) of responsive extension, nd by tems vi) of responsive extension $\mu'' P_U \mu$, which implies that for transitive property of R_U, $\mu' P_U \mu$.

Then the matching μ is q -blocked for coalition S and $\mu \notin C(M_U^q)$.

We define the set $\overline{T}_{<q}(M) = T_{<q}(M) \setminus R$ and we will prove that it is a subseteq of q -core.

Proposition 4 *If* $M_U^q = (M, R_U, q)$ *is a matching market with quota restriction, then* $\overline{T}_{<q}(M) \subseteq C(M_U^q)$.

The proof to this proposition is developed in the Appendix as Proposition A8.

4 The *q*-core and the *q*-stable

The following result establishes that every matching in the q -core is also a q -stable of M_U^q.

Proposition 5 *If* $M_U^q = (M, R_U)$ *is a matching market with quota restriction, then* $C(M_U^q) \subseteq S(M_U^q)$.

Proof. Assume that $\mu \in C(M_U^q)$ and $\mu \notin S(M_U^q)$.

From $\mu \in C(M_U^q)$, by proposition 3, μ is q -individually rational. As $\mu \notin S(M_U^q)$, then there exists (d,e) that q -blocked to μ.

Let $S=\{d,e\}$ and let the matching $\mu' = \mu_{(d,e)}$, where

$$\mu'(x)=\begin{cases}\mu(x) & if \quad x\notin\{d,e,\mu(d),\mu(e)\\ d & if \quad x=e\\ e & if \quad x=d\\ \phi & otherwise.\end{cases}$$

Conditions 1) and 3) of q -blocked by coalition are satisfied by definition of μ'. As μ'(d)=e, condition 1) of q -blocked by pair implies conditions 2) of q -blocked by coalition.

To prove Condition 4), we consider μ(d)=∅ or μ(e)=∅, then Conditions 2) of q -blocked by coalition imply that $\mu' R_U \mu$; which contradicts that $\mu \in C(M_U^q)$.

The results obtained until now allow us to give a complete characterization of the q -core in terms of sets $T_q(M)$ and $\overline{T}_{<q}(M)$.

Theorem 1 *If $M_U^q=(M,R_U,q)$ is a matching market with quota restriction, then* $C(M_U^q)=T_q(M)\cup\overline{T}_{<q}(M)$.

Proof . From Proposition 4 and Proposition 6 we obtain:

$$T_q(M)\cup\overline{T}_{<q}(M)\subseteq C\ (M_U^q). \qquad (1)$$

Let $\mu\in C_q(M_U^q)$. Then, by Proposition 5, $\mu\in S(M_U^q)$. As $S(M_U^q)=T_q(M)\cup T_{<q}(M)$, we obtain:

$$\mu\in T_q(M) \text{ or } \mu\in\overline{T}_{<q}(M). \qquad (2)$$

If #μ=q[1], by (2), we have that $\mu \in T_q(M)$.

Let #μ<q and let us assume that $\mu\notin\overline{T}_{<q}(M)$, which implies, by (2), that $\mu\in T_{<q}(M)$ and $\mu\notin T_{<q}(M)\setminus R$; then $\mu \in R$ and, by Proposition 5, $\mu\notin C(M_U^q)$, which contradicts $\mu\in C(M_U^q)$. We obtain:

$$C(M_U^q)\subseteq T_q(M)\cup\overline{T}_{<q}(M). \qquad (3)$$

(1) and (3) imply that $C(M_U^q)=T_q(M)\cup\overline{T}_{<q}(M)$.

5 Existence of the q-core

In order to guarantee that $C(M_U^q)\neq\varnothing$, we prove that, given (M_U^q) a model of market, sets $T_q(M)$ and $T_{<q}(M)$ are not empty simultaneously.

Lemma 1 *If $T_q(M)\neq\varnothing$, then $T_{\tilde q}(M)\neq\varnothing$, for every $\tilde q<q$.*

Proof. Let $\mu \in T_q(M)$, which implies that there exists $(t_1,t_2) \in N$, such that $\mu\in S(M^{(t_1,t_2)})$ and #μ=q.

Without loss of generality, we assume that $t_1=max\{t:\mu(d_t)\neq\varnothing\}$, which implies that $\mu(d_{t_1})\neq\varnothing$.

Since $S(M^{(t_1-1,t_2)})\neq\varnothing$, we have that $\tilde\mu\in S(M^{(t_1-1,t_2)})$. Since μ and $\tilde\mu$ are stable in $M^{(t_1,t_2)}$ and $M^{(t_1-1,t_2)}$, respectively, by Lemma A1 in Appendix, $\#\tilde\mu\leq\#\mu\leq\#\tilde\mu+1$, which implies that $\#\tilde\mu=q-1$ o $\#\tilde\mu=q$.

If $\#\tilde\mu=q-1$, then $\tilde\mu\in T_{q-1}(M)$ and, therefore, $T_{q-1}(M)\neq\varnothing$. If $\#\tilde\mu=q$, there exists $\tilde{\tilde\mu}\in S(M^{(t_1-2,t_2)})$, such that $\#\tilde{\tilde\mu}\leq\#\tilde\mu\leq\#\tilde{\tilde\mu}+1$, then $\#\tilde{\tilde\mu}=q-1$ o $\#\tilde{\tilde\mu}=q$

[1] We denote the cardinality of a matching μ by μ $\#\mu=\#\{d:\mu(d)\in E\}=\#\{e:\mu(e)\in D\}$.

Since t_1 is finite, there exists k and $\tilde{\mu} \in S(M^{(t_1-k,t_2)})$, such that $\#\tilde{\mu} = q-1$, then $T_{q-1}(M) \neq \varnothing$. By repeating this process the result that follows is obtained: $T_{\tilde{q}}(M) \neq \varnothing$, for every $\tilde{q} < q$.

Let us prove now that set $C(M_U^q)$ is not empty.

Theorem 2 *If* $M_U^q = (M; R_U, q)$. *is a matching market with quota restriction, then* $C(M_U^q) \neq \varnothing$

Proof. Let $T_q(M) \neq \varnothing$, by Proposition 2, $C(M_U^q) \neq \varnothing$.

Let us assume $T_q(M) \neq \varnothing$ and let $\mu \in S(M)$ and $\#\mu = \bar{q}$. Notice that $\bar{q} < q$ and $T_{\bar{q}}(M) = S(M) \neq \varnothing$, then

$$C(M_U^{q'}) \neq \varnothing \text{ for every } q' \leq \bar{q}. \tag{4}$$

Now, we will show that:

$$C(M) \subseteq C(M_U^{q'}) \text{ for } q' > \bar{q}. \tag{5}$$

Suppose that $\mu \in C(M)$ y $\mu \notin C(M_U^{q'})$, with $q' > \bar{q}$.

Since $\mu \in C(M) = S(M) = T_{\bar{q}}(M)$, then $\mu \in T_{\bar{q}}(M)$, which implies that $\#\mu = \bar{q} < q'$ and for every coalition $S \subseteq E \cup D$, there is no matching $\mu' \in \mathrm{M}$ that satisfies Conditions 1. y 2. of *q'*-blocked by coalition. This contradicts $\mu \notin C(M_U^{q'})$.

By (5) and $C(M_U^q) \neq \varnothing$, it is implied that

$$C(M_U^{q'}) \neq \varnothing \text{ for every } q' > \bar{q}. \tag{6}$$

From (4) and (6) we conclude that:

$$C(M_U^q) \neq \varnothing \text{ for every } q. \tag{7}$$

6 Conclusions

In this research work the concept of core is defined in the matching model with quota restriction and is called q-core. For this purpose the institution's role in the concept of blocking by coalition in the bilateral assignation model was reconsidered.

The existence of the q-core under the institution's responsive preference restriction is proved. By means of Example 1 it is stated that, unlike what happens in the bilateral model, the q-stables and the q-core may not coincide; nevertheless, Theorem 1 shows how a characterization of the q-core can be obtained in terms of subsets of the set of q-stables.

Appendix

The restriction of *M*

From now on, we will denote $F \in \{D, E\}$ and $F^c \in \{D, E\}$ such that $\{F, F^c\} = \{D, E\}$ and $f \in F$ will denote a generic worker.

Given $F' \subseteq F$, we denote the restriction of P_F to F' by $P_{|F'}$. Given $M = (F, F^C, \mathbf{P})$, we denote the restriction of M to F' by $M_{F'} = (F', F^C, P_{|F'}, P_{F^C})$. For the sake of simplicity we denote $M_{F'} = (F', F^C, \mathbf{P})$, where we have to understand that $\mathbf{P} = (P_{|F'}, P_{F^C})$.

Lemma A1 *(Femenia, Marí, Oviedo and Neme 2008) Given M=(D,E,P) and* $F' \subseteq F$, *let* μ *and* μ' *be the stable matchings for M and* $M_{F'}$ *respectively. Then* $\#\mu' \leq \#\mu \leq \#\mu' + \#(F \setminus F')$.

The institution's responsive preference

Given a matching market M_U and a quota $q \leq \min\{n,m\}$, we denote A $S(M_U^q)$ the set of all q -stable matchings. We will assume that the institution has an individual preference $\succ_D$ over the set $D \cup \varnothing$ and an individual preference $\succ_E$ over the set $E \cup \varnothing$ and its preference over matchings are directly connected with its preferences over workers. An institution's preference is called responsive to its individual preferences if, for any matching that differs in only one worker, the institution prefers the matching that has the most preferable worker according to the individual preferences.

In order to formalize the institution's responsive preference, we introduce the notations that follow.

For every matching μ, consider $B_\mu = \{(d,e) \in D \times E : \mu(d) = e\}$.

For every $f \in D \cup E$:

$$\mu^{(d,e)}(f) = \begin{cases} \varnothing & si \quad f \notin \{d,e\} \\ d & if \quad f = e \\ e & if \quad f = d. \end{cases}$$

Notice that $\mu^{(d,e)} = \mu^{\varnothing}_{(d,e)}$.

Definition A2 *A preference relation* R_U *is a* ***responsive extension*** *of preferences* $\succ_D$ *and* $\succ_E$ *over* $D \cup \varnothing$ *and* $E \cup \varnothing$ *respectively, such that it satisfies the following conditions:*

i) $\mu^{(d,e)} P_U \mu^{\varnothing}$ if and only if $d \succ_D \varnothing$ and $e \succ_E \varnothing$.

ii) $\mu P_U \mu^{\varnothing}$ if and only if $\mu^{(d,e)} P_U \mu^{\varnothing}$ for every $(d,e) \in B_\mu$.

iii) $\mu^{(d,e)} P_U \mu^{(d,e')}$ if and only if $e \succ_E e'$.

iv) $\mu^{(d,e)} P_U \mu^{(d',e)}$ if and only if $d \succ_D d'$.

v) For every $\mu,\ \mu' \in \mathsf{M}$ such that $\#\mu = \#\mu'$ and $B_\mu = B_{\mu'} \setminus \{(d',e')\} \cup \{(d,e)\}$:
$\mu P_U \mu'$ if and only if $\mu^{(d,e)} P_U \mu^{(d',e')}$

vi) For every $\mu,\ \mu' \in \mathsf{M}$ such that $B_{\mu'} \subset B_\mu$ and $\mu P_U \mu^{\varnothing}$, then $\mu P_U \mu'$.

vii) For every $\mu,\ \mu' \in \mathsf{M}$ such that $\mu(E) = \mu'(E)$ and $\mu(D) = \mu'(D)$, then $\mu I_U \mu'$.

We consider a preference R_U to be responsive if there are two individual preferences $\succ_D$ and $\succ_E$ over $D \cup \{\varnothing\}$ and $E \cup \{\varnothing\}$ respectively, such that R_U is a responsive extension.

Remark A3 *Given two preferences* $\succ_D$ *and* $\succ_E$, *over* $D \cup \{\varnothing\}$ *and* $E \cup \{\varnothing\}$ *respectively, we can construct a responsive preference relation* R_U *over the set of all matchings* M*; moreover, this extension is not unique.*

The sets $T_q(M)$ and $T_{<q}(M)$

Now we will consider the model M_U^q, where R_U is a responsive preference. Without loss of generality and in order to avoid the addition of notational complexity to the model M_U^q, we assume that all the agents of sets D and E are acceptable for the institution, i.e. for every $d \in D$ and $e \in E$, we have that $d \succ_D \varnothing$ and $e \succ_E \varnothing$.

For every $t \in \mathsf{N}$, we can define the following subset $F^t \subseteq F$ such that $\#F^t = t$, and for every $f \in F^t$ and $f' \notin F^t$ we have that $f \succ_F f'$. Note that $F^1 \subseteq F^2 \subseteq ... \subseteq F^l = F$, where $\#F = l$.

Given sets $\mathbf{d} = \{1,2,...,\#D\}$ and $\mathbf{e} = \{1,2,...,\#E\}$; for every $(t_1,t_2) \in \mathbf{d} \times \mathbf{e}$, we denote $M^{(t_1,t_2)}$, the restriction of M to D^{t_1} and E^{t_2}, i.e., $M^{(t_1,t_2)} = \left(D^{t_1}, E^{t_2}, \mathbf{P}\right)$.

Given $(t_1,t_2)\in \mathbf{d}\times\mathbf{e}$ q, and the following sets of matchings:

$$T_q(M^{(t_1,t_2)}) = \begin{cases} S(M^{(t_1,t_2)}) & \text{if } \#\mu = q \text{ for every } \mu\in S(M^{(t_1,t_2)}) \\ \varnothing & \text{otherwise} \end{cases}$$

and

$$T_q(M) = \{\mu : \exists \quad (t_1,t_2) \text{ such that } \mu\in T_q(M^{(t_1,t_2)})\}.$$

Proposition A4 *Given* $M_U=(M,R_U)$, $(t_1,t_2)\in\mathbf{d}\times\mathbf{e}$ *there exists* $K\subseteq\mathbf{d}\times\mathbf{e}$*, such that*

$$T_q(M) = \dot{\bigcup_{(t_1,t_2)\in K}} T_q(M^{(t_1,t_2)}).$$

Given $(t_1,t_2)\in\mathbf{d}\times\mathbf{e}$, q and the following sets of stable matchings:

$$T_{<q}(M^{(t_1,t_2)}) = \left\{ \mu\in S(M^{(t_1,t_2)}) : \#\mu<q, \text{either } \varnothing P_e d \text{ or } \varnothing P_d e \text{ for every } d\in D\setminus\mu(E^{t_1}) \text{ and } e\in E\setminus\mu(D^{t_2}) \right\}$$

and

$$T_{<q}(M) = \{\mu : \exists \quad (t_1,t_2) \text{ such that } \mu\in T_{<q}(M^{(t_1,t_2)})\}.$$

Proposition A5 *Given* $M_U=(M,R_U)$, $(t_1,t_2)\in\mathbf{d}\times\mathbf{e}$ *there exists* $\hat{K}\subseteq\mathbf{d}\times\mathbf{e}$*, such that*

$$T_{<q}(M) = \dot{\bigcup_{(t_1,t_2)\in\hat{K}}} T_{<q}(M^{(t_1,t_2)}).$$

Remark A6 *The sets* K *and* $\hat{K}$ *on the previous propositions are given by:*

$K=\{(t_1,t_2)\in N : \forall\,(t_1',t_2')\neq(t_1,t_2) \quad t_1'\le t_1, t_2'\le t_2$ such that

$T_q(M^{(t_1,t_2)})\cap T_q(M^{(t_1',t_2')})=\varnothing\}$ and

$\hat{K}=\{(t_1,t_2)\in N : \forall\,(t_1',t_2')\neq(t_1,t_2) \quad t_1'\le t_1\ t_2'\le t_2$ such that

$T_{<q}(M^{(t_1,t_2)})\cap T_{<q}(M^{(t_1',t_2')})=\varnothing\}$.

Proposition A7 If $M_U^q=(M,R_U,q)$ is a matching market with quota restriction, then $T_q(M)\subseteq C(M_U^q)$.

Proof. We assume that $\mu\in T_q(M)$ and; then $\#\mu=q$ and there exist a coalition $S\subseteq D\cup E$ and a matching $\mu'\in\mathsf{M}_q$ that satisfy the conditions of q-blocked by coalition.

Notice that if μ is q-blocked by the coalition S, then μ is blocked by the coalition S. Since $\mu\in T_q(M)$, there exists $(t_1,t_2)\in N$, such that $\mu\in S(M^{(t_1,t_2)})$ and, as $C(M)=S(M)$, $S(M^{(t_1,t_2)})=C(M^{(t_1,t_2)})$, then μ is not q-blocked by any coalition $S\subseteq D^{t_1}\cup E^{t_2}$; thus $S\not\subseteq D^{t_1}\cup E^{t_2}$, which implies that

there exists $x\in S$ such that $\mu(x)=\varnothing$. (8)

Since $S\subseteq D\cup E$, we can write $S=S_D\cup S_E$, there being $S_D=S\cap D$ and $S_E=S\cap E$. We consider the following sets:

$$S_D^1=S_D\cap D^{t_1},\quad S_E^1=S_E\cap E^{t_2},$$
$$S_D^2=S_D\setminus D^{t_1},\quad S_E^2=S_E\setminus E^{t_2}. \qquad (9)$$

We can write S_D y S_E:

$$S_D=S_D^1\,\dot{\bigcup}\,S_D^2,\ S_E=S_E^1\,\dot{\bigcup}\,S_E^2. \qquad (10)$$

Now, we will show that

$$\mu^{'}(S_E^1) \subseteq S_D^2 \text{ y } \mu^{'}(S_D^1) \subseteq S_E^2. \tag{11}$$

Let us assume that $e \in S_E^1$ and let us consider $\mu^{'}(e) \notin S_D^2$; then $\mu^{'}(e) \in D^{t_1}$. However, $e \in S$ and S is a coalition that blocks μ, then we have that $d = \mu^{'}(e) P_e \mu(e)$ and $e = \mu^{'}(d) P_d \mu(d)$, which implies that $(d,e) \in D^{t_1} \times E^{t_2}$ is a q -bloching pair of μ. This contradicts $\mu \in S(M^{(t_1,t_2)})$.

Similary, we can prove that $\mu^{'}(S_D^1) \subseteq S_E^2$.

By (11) and Condition 1. of q -blocked by coalition we obtain

$$\# S_E^1 = \# \mu^{'}(S_E^1) \leq \# S_D^2 \text{ and } \# S_D^1 = \# \mu^{'}(S_D^1) \leq \# S_E^2 \tag{12}$$

Let $D^{t^1} = (D^{t^1} \setminus S_D^1) \dot{\bigcup} S_D^1$ and

$$D^{t_1} \setminus S_D^1 = D^{t_1} \setminus (S_D^1 \cup \mu(S_E^1)) \dot{\bigcup} \mu(S_E^1) \setminus S_D^1, \tag{13}$$

We express #μ y #μ' in terms of the sets defined in (9).

$$\# \mu = \# \{ d \in D^{t_1} \setminus (S_D^1 \cup \mu(S_E^1)) : \mu(d) \neq \varnothing \} +$$
$$\# \mu(S_E^1) \setminus S_D^1 + \# \{ d \in S_D^1 : \mu(d) \neq \varnothing \}. \tag{14}$$

Let now,

$$\mu^{'}(E) = \{ d \in D \setminus S_D : \mu^{'}(d) \neq \varnothing \} \dot{\bigcup} \{ d \in S_D : \mu^{'}(d) \neq \varnothing \}. \tag{15}$$

By condition 1. of q -blocked by coalition we have that $\{ d \in S_D : \mu^{'}(d) \neq \varnothing \} = S_D$ and, by (10), it is implied that $\{ d \in S_D : \mu^{'}(d) \neq \varnothing \} = S_D^1 \dot{\bigcup} S_D^2$. From $D^{t_1} \setminus (S_D^1 \cup \mu(S_E^1)) \subset D \setminus S_D$ and (15) we can find a set A such that

$$\# \mu^{'}(E) = \# \{ d \in D^{t_1} \setminus (S_D^1 \cup \mu(S_E^1)) : \mu^{'}(d) \neq \varnothing \} + \# A + \# S_D^1 + \# S_D^2, \tag{16}$$

and by Condition 3. of q -blocked by coalition we have that:

$$\# \mu^{'}(E) = \# \{ d \in D^{t_1} \setminus (S_D^1 \cup \mu(S_E^1)) : \mu(d) \neq \varnothing \} + \# A + \# S_D^1 + \# S_D^2. \tag{17}$$

As $\# \mu = q$ and $\# \mu^{'} \leq q$ and, by (14) and (17), we have that:

$$\# A + \# S_D^1 + \# S_D^2 \leq \# \{ d \in S_D^1 : \mu(d) \neq \varnothing \} + \# \mu(S_E^1) \setminus S_D^1. \tag{18}$$

By $\{ d \in S_D^1 : \mu(d) \neq \varnothing \} \subseteq S_D^1$,(11), (17) and (18) we have that:

$$\# S_D^1 + \# S_D^2 \leq \# S_D^1 + \# S_D^2,$$

which implies that:

$$\{ d \in S_D^1 : \mu(d) \neq \varnothing \} = S_D^1, \# \mu(S_D^1) \setminus S_E^1 = \# \mu(S_D^1) = \# S_D^2, \mu(S_E^1) \setminus S_D^1 = \mu(S_E^1),$$

and $\mu(S_E^1) \cap S_D^1 = \varnothing$.

Notice that, for every $d \in S_D^1$, $\mu(d) \neq \varnothing$, $\mu(d) \notin S_E^1$ and $\mu(\mu^{'}(d)) = \varnothing$; this implies that we can define the following sequence of matchings $\mu = \mu_0 \mu_1 \ldots \mu_r$, such that $\mu_k = \mu_{k-1(d,\mu^{'}(d))}$.

Since $\mu_k = \mu_{k-1(d,\mu^{'}(d))}$, note that, as $\mu(d) \neq \varnothing$ and $\mu(\mu^{'}(d)) = \varnothing$ and this implies that μ_k is definite well, $\# \mu_k = \# \mu_{k-1}$ and $B_{\mu_k} = B_{\mu_{k-1}} \setminus \{ (d, \mu(d)) \} \cup \{ (d, \mu^{'}(d) \}$.

As R_U is a responsive extension, $\mu_{k-1} P_U \mu_k$ and, by transitivity of R_U,

$$\mu P_U \mu_r. \tag{19}$$

Symmetrically, by considering $\# \mu = \mu(D)$ y $\# \mu^{'} = \mu^{'}(D)$, we obtain $\# \mu(S_E^1) \setminus S_D^1 = \# \mu(S_E^1) = \# S_E^2, \{ e \in S_E^1 : \mu(e) \neq \varnothing \} = S_E^1, \mu(S_D^1) \setminus S_E^1 = \mu(S_D^1)$ and $\mu(S_D^1) \cap S_E^1 = \varnothing$.

This implies that, for every $e \in S_E^1$, $\mu(e) \neq \varnothing$, $\mu(e) \notin S_D^1$ and $\mu(\mu^{'}(e)) = \varnothing$ and now we can define the sequence of matching $\mu_r \mu_{r+1} \ldots \mu^{'}$, such that $\mu_k = \mu_{k-1(\mu^{'}(e),e)}$. By transitivity of R_U we have that

$$\mu_r P_U \mu. \tag{20}$$

By (19), (20) and transitivity of R_U,

$\mu P_U \mu'$. (21)

Since, by (8), there exists $x \in S$ such that $\mu(x) = \varnothing$, Condition (21) contradicts the fact that μ is q-blocked by coalition S. Finally $\mu \in C(M_U^q)$.

Proposition A8 If $M_U^q = (M, R_U, q)$ is a matching market with quota restriction, then $\overline{T}_{<q}(M) \subseteq C(M_U^q)$.

Proof. We Assume that $\mu \in \overline{T}_{<q}(M)$ and $\mu \notin C(M_U^q)$ then $\#\mu < q$ and there exist a coalition $S \subseteq D \cup E$ and a matching $\mu' \in \mathsf{M}_q$ that satisfy the conditions of q-blocked by coalition

Notice that if μ is q-blocked by the coalition S, then μ is blocked by the coalition S. Since $\mu \in T_{<q}(M)$, there exists $(t_1, t_2) \in N$, such that $\mu \in S(M^{(t_1,t_2)})$ and, as $C(M) = S(M)$ $S(M^{(t_1,t_2)}) = C(M^{(t_1,t_2)})$, then μ is not q-blocked by any coalition $S \subseteq D^{t_1} \cup E^{t_2}$; thus $S \not\subseteq D^{t_1} \cup E^{t_2}$, which implies that

there exists $x \in S$ suchthat $\mu(x) = \varnothing$. (22)

Since $S \subseteq D \cup E$, we can write $S = S_D \cup S_E$, there being $S_D = S \cap D$ and $S_E = S \cap E$. We consider the following sets:

$$\begin{aligned} &S_D^1 = S_D \cap D^{t_1}, \quad S_E^1 = S_E \cap E^{t_2}, \\ &S_D^2 = S_D \setminus D^{t_1}, \quad S_E^2 = S_E \setminus E^{t_2}. \end{aligned} \tag{23}$$

We have that S_D y S_E:

$$S_D = S_D^1 \dot{\bigcup} S_D^2, S_E = S_E^1 \dot{\bigcup} S_E^2. \tag{24}$$

And

Now, we will show that

$S_D^1 \setminus \mu(E) = \varnothing$ and $S_E^1 \setminus \mu(D) = \varnothing$. (25)

We assume that there exists $d \in S_D^1 \setminus \mu(E)$. By condition 1. of q-blocked by coalition, $\mu'(d) = e \in S_E$, which imples that $e \in S_E^1$ or $e \in S_E^2$.

Let us assumethat $e \in S_E^1$, then $e \in S_E^2$. As $e \in S$ and S is a coalition that blocks μ, we have that $d = \mu'(e) P_e \mu(e)$ and $e = \mu'(d) P_d \mu(d)$, which implies that $(d,e) \in D^{t_1} \times E^{t_2}$ is a q-blocking pair of μ. It contradicts $\mu \in S(M^{(t_1,t_2)})$.

If $e \in S_E^2$, then $\mu(e) = \varnothing$, and, since $d \in S_D^1 \setminus \mu(E)$, $\mu(d) = \varnothing$. As S is a coalition that blocks μ, we have that $d = \mu'(e) P_e \mu(e)$ and $e = \mu'(d) P_d \mu(d)$, which implies that $(d,e) \in D \setminus \mu(E^{t_2}) \times E \setminus \mu(D^{t_1})$. It contradicts $\mu \in T_{<q}(M)$. Then $S_D^1 \setminus \mu(E) = \varnothing$.

Similary, we can prove that $S_E^1 \setminus \mu(D) = \varnothing$.

Now, we show that

$\mu'(S_E^1) = S_D^2$ y $\mu'(S_D^1) = S_E^2$. (26)

Let us assume that $e \in S_E^1$ and $\mu'(e) \notin S_D^2$. By condition 1. of q-blocks by coalition, $\mu'(e) \in S_D^1 \subseteq D^{t_1}$. Since $e \in S$, $d = \mu'(e) P_e \mu(e)$ y and $e = \mu'(d) P_d \mu(d)$, then $(d,e) \in D^{t_1} \times E^{t_2}$ is a blocking pair of μ, which contradicts $\mu \in S(M^{(t_1,t_2)})$.

Similary, we can prove that $\mu'(S_D^1) \subseteq S_E^2$.

Since $\mu \in T_{<q}(M)$, for every $d \in S_D^2$ and for every $e \in S_E^2$, d and e are not mutually acceptable; then $\mu'(d) \notin S_E^2$ and $\mu'(e) \notin S_D^2$; by Condition 1. of q-blocks by coalition, $\mu'(d) \in S_E^1$ and $\mu'(e) \in S_D^1$, with which the proof to (26) is completed .

Since $\mu \notin R$, for every $d \in S_D^1$, $\mu(d) \notin S_E^1$ and, for every $e \in S_E^1$, $\mu(e) \notin S_D^1$, then:

$S_D^1 \cap \mu(S_E^1) = \varnothing$ y $S_D^1 \cap \mu(S_E^1) = \varnothing$. (27)

By (26) and Condition 1. of q -blocks by coalition, $\#\mu = \#\mu'$. By (25) and (27), for every $d \in S_D^1$, $\mu(d) \neq \emptyset$ and $\mu(d) \notin S_E^1$. Besides, we have that, by Condition 1. of q -blocks by coalition, $\mu'(d) \in S_E^2$ and, consequently, $\mu(\mu'(d)) = \emptyset$. This implies that we can define the following sequence of matchings $\mu = \mu_0 \mu_1 ... \mu_r$, such that $\mu_k = \mu_{k-1(d,\mu'(d))}$.

Since $\mu_k = \mu_{k-1(d,\mu'(d))}$, note that, as $\mu(d) \neq \emptyset$ and $\mu(\mu'(d)) = \emptyset$, it is implied that μ_k is a matching such that, $\#\mu_k = \#\mu_{k-1}$ and $B_{\mu_k} = B_{\mu_{k-1}} \setminus \{(d, \mu(d)\} \cup \{(d, \mu'(d)\}$.

As R_U is a responsive extension, $\mu_{k-1} P_U \mu_k$ and, by transitivity of R_U,

$$\mu P_U \mu_r. \qquad (28)$$

Symmetrically, by considering $S_E^1 \setminus \mu(D) = \emptyset$, $\mu(S_E^1) \cap S_D^1 = \emptyset$ and by Condition 1. of q -blocked by coalition, we have that for every $e \in S_E^1$,

$$\mu(e) \neq \emptyset, \mu(e) \notin S_D^1 \text{ y } \mu(\mu'(e)) = \emptyset$$

which implies that we can define the sequence of matching $\mu_r \mu_{r+1} ... \mu'$, such that $\mu_k = \mu_{k-1(\mu'(e),e)}$. By transitivity of R_U we have that

$$\mu_r P_U \mu. \qquad (29)$$

By (28), (29) and transitivitye of R_U,

$$\mu P_U \mu'. \qquad (30)$$

Since by (22) there exists $x \in S$ such that $\mu(x) = \emptyset$, condition (30) contradicts the fact that μ is q -blocked by coalition S. Finally, $\mu \in C(M_U^q)$.

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