The Fenchel Duality Theorem and the Negishi Approach[1]

Elvio Accinelli[2], Leobardo Plata[3] and Martín Puchet [4]

[2] UASLP, Facultad de Economía, San Luis Potosí- México E-mail adress: elvio@correo.xoc.uam.mx
[3] UASLP, Facultad de Economía, San Luis Potosí- México E-mail adress: lplata@uaslp.mx
[4] UNAM, Facultad de Economía, México D.F. E-mail adress: anyul@servidor.unam.mx

Resumo: Neste trabalho analisamos a relação entre o tratamento de Negishi e o teorema de dualidade de Fenchel.

Palavras - chaves: atualidade, teorema de Fenchel, tratamento de Negishi.

Abstract: In this work we analyze the relation between the Negishi approach and the Fenchel's duality theorem.

Keywords: *Duality, Fenchel's theorem, Negishi approach.*
JEL Classification: C61, D60

[1] Our research is supported by Conacyt - México project number 46209 ``Modelado matemático del cambio en economía''

Introduction

The Negsihi approach is a good tool to characterize the set of Pareto optimal allocations and to generalize this characterization to economies with infinitely many goods and a finite set of agents. Using this approach, we transform an infinite dimensional optimization problem related with Pareto optimal allocations into a finite dimensional problem related with agents and their relative weights in the market. Feasible allocations $x \in F$ are represented by vectors $x = (x_1, ..., x_n)$ in the cartesian product of n vectorial spaces X, satisfying the inequalities $\sum_{i=1}^{n} x_i \leq W$ where W is the vector representing the aggregate wealth of the economy. If the economy has infinitely many goods, then X is an infinite dimensional vectorial space. The starting point for this work is the problem of the maximization of a social utility function. The weights of the agents in this social utility function are characterized by a vector $\lambda \in \Delta$, (with coordinates representing their relative weights in a social utility function) in the $n-$ dimensional simplex Δ, where n is the number of agents in the economy and this is a finite number. As it is well known, for each λ^* in the simplex there exists a Pareto optimal allocation x^* such that this allocation solves the problem of maximizing the social utility $U_{\lambda^*}(x) = \sum_{i=1}^{n} \lambda_i^* u_i(x_i)$, *s.t.* $x \in F$. And reciprocally, for each x^* Pareto optimal allocation, there exists a vector $\lambda \in \Delta$, such that $U_{\lambda^*}(x^*) \geq U_{\lambda^*}(x)$, $\forall x \in F$. Then, choosing choose a Pareto optimal allocation $x^* \in F$ satisfying given conditions of welfare, is equivalent to choose certain $\lambda \in \Delta$, considering the corresponding (''dual'') conditions. In this paper, we will consider the problem of maximizing this welfare social utility as a direct problem, and the dual problem of this maximization problem. In other words, we transform the problem of choosing an allocation maximizing a social utility function restrict to a subset of an infinite dimensional space, into the problem of choosing a vector $\lambda^* \in \Delta$, minimizing a finite dimensional dual problem.

The Fenchel's theorem is a good tool to solve optimization problems in finite or infinite dimensional spaces, when it is possible to introduce a dual space of the original one. The Negishi approach transforms our main infinite dimensional problem into a finite dimensional one. To solve our optimization problem the Fenchel's duality theorem appears to be natural and moreover, using this theorem and the Negishi approach, we obtain a direct and deeper insight into the problem of characterizing the Pareto allocation that maximizes the social welfare.

The dual problem

Definition *Let X be a normed space. The space of all bounded linear functionals on X is called* **the normed dual** *of X and is denoted by x^*.*

Definition *Let f be a convex functional defined on a convex set C in a normed space X. The* **conjugate set** *C^* is defined as*

$$C^* = \left\{ y^* \in X^* : \sup_{y \in C} [yy^* - f(y) < \infty] \right\},$$

and the functional f^* conjugate to f is defined on C^* as:

$$f^*(y^*) = \sup_{y \in C} [yy^* - f(y)].$$

We will denote **the epigraph** of f over C by $[f, C]$.

To develop a geometric interpretation of the conjugate functional as k varies, note that the solution (r, y) of the equation

$$yy^* - r = k$$

describes a parallel closed hyperplane in $R \times X$. The number $f^*(y)$ is the supremum of the values of k for which the hyperplane intersects $[f, C]$. So, $yy^* - r = f^*(y)$ is a support hyperplane of $[f, C]$.

Theorem (The Fenchel's duality theorem) *Assume that f and g are, respectively, convex and concave functionals on the convex sets C and D in a normed space X. Assume that $C \cap D$ contains points in the relative interior of C and D and either [f,C] or [g,D] has nonempty interior. Suppose further that $\mu = inf_{y \in C \cap D} \{f(y) - g(y)\}$ is finite. Then*

$$\mu = \inf_{y \in C \cap D} \{f(y) - g(y)\} = \max_{y^* \in C^* \cap D^*} \{g^*(y^*) - f^*(y^*)\}$$

where the maximum on the right is achieved by some $y_0^* \in C^* \cap D^*$.
This theorem is shown in [2].

The Negishi dual approach

Let $U = \{(u_1,...,u_n) \in R^n$*:thereisafeasibleallocation* $x : u_i \leq u_i(x_i)$ *for* $i = 1,...,n\}$ be *the utility possibility set,* and let UP be the Pareto frontier. This set is the north-east boundary of U Let u be the profile of utilities, $u = (u_1,...,u_n)$. See [3].

For each vector $\lambda = (\lambda_1,...,\lambda_n)$ fixed, consider the following maximization problem:

$$\max_{u\in U} \sum_{i=1}^{n} \lambda_i u_i. \tag{1}$$

To see the economical intuition of this problem, see 4. Note that this problem is equivalent to the following one: $\max_{x\in F} \sum_{i=1}^{n} \lambda_i u_i(x_i)$, where F is the feasible set, see [1].

To solve this problems by means of the Fenchel's theorem, we note that our problem is obtaining de maximum of a concave function $\Psi_\lambda(u) = \sum_{i=1}^{n} \lambda_i u_i$ restrict to a convex set U. Them, the dual problem will be to minimize the corresponding dual functional. We consider the set D to the positive orthant, accordingly we take $C = \{u \in U\}$ and $g(u) \equiv 0$. Them, the functional conjugate to g is:

$$g^*(\nu) = \sup_{u\in U} u\nu \quad and \quad C^* = \left\{\nu : \sup_{u\in U} u\nu < \infty\right\}$$

From the characteristics of the problem we consider only $\mu \in \Delta$. The solution of this problem is given by $u^* = (u_1^*,...,u_n^*)$, a profile in the Pareto frontier.

Consider de conjugate function $f^*(\nu) = \inf_{u\in R_+^n}[u\nu - \sum_{i=1}^{n} \lambda_i u_i]$. Now for each i we define the functional

$$f_i^*(\nu_i) = \inf_{u_i \in R+} [u_i \nu_i - \lambda_i u_i]$$

The solution of this problem is $u_i = 0;\ \forall\, i = 1,...,n$. It is clear that $f^*(\nu) = \sum_{i=1}^{n} f_i^*(\nu_i)$.

Now the equality $\sup_{u\in C\cap D}\{f(u) - g(u)\} = \min_{\nu\in C^*\cap D^*}\{g^*(\nu) - f^*(\nu)\}$ takes the form:

$$\sup_{u\in U}\left\{\sum_{i=1}^{n} \lambda_i u_i\right\} = \inf_{\mu\in C^*\cap D^*}\left\{\sum_{i=1}^{n} \nu_i u_i\right\},$$

where $\sum_{i=1}^{n} \nu_i u_i = g^*(\nu)$ and $C^* \cap D^* = \{\mu \in R_+^n\}$ However, from the characteristics of this problem we can restrict ourselves to choose $\nu \in \Delta$.

Observe that $U_{\bar\lambda} = \max_{u\in U} \sum_{i=1}^{n} \bar\lambda_i u_i$ for $\bar\lambda$ given, is equal to $U_{\bar u} = \min_{\lambda\in\Delta} \sum_{i=1}^{n} \lambda_i \bar u_i$ where u is the solution of the previous maximization problem given in (1) so, $\bar u \in UP$.

Now, calculating

$$\min_{\lambda\in\Delta} \sum_{i=1}^{n} \lambda_i u_i(x_i^*(\lambda)) \tag{2}$$

we obtain the maximal possible value of U_λ for each $u \in UP$. Because, moving $\lambda \in \Delta$, the function $u^* = u(x^*(\lambda))$ is taking all the possible values of u in UP thus solving (2) and taking account the Fenchel's duality theorem, we obtain the maximum possible value of the social utility in the subset of all feasible Pareto optimal allocations. Note that using continuity arguments the solution for problem (2) exists.

Bibliographical Reference

Accinelli, E. 1994,``*Some Remarks about Uniqueness of the Equilibrium for Infinite Dimensional Economies.´´* Estudios Económicos **21,** , pp. 315-326.

Luenberger, D. 1969, Optimization by vector space methods. *John Wiley and Sons ltd.*

Mas-Colell, A. Whinston, M.; Green, J. 1995, Microeconomic Theory' *Oxford University Press* .

Negishi, T. 1960, ``*Welfare Economics and Existence of an Equilibrium for a Competitive Economy´´ Metroeconomica* **12**, 92- 97.