INFERENCE ON INSTRUMENTAL VARIABLES MODELS: A NOTE FOR PRACTITIONERS

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Abstract - When using instrumental variables (IV) methods, practitioners learn that two types of residuals may be obtained. We collect a few results from these types of residuals and clarify the use of a goodness of fit measure by Pesaran and Smith (Econometrica, 1994, p.705-710) that may be useful to complement inference on IV estimated regression models.

Key-words: *instrumental variables, goodness of fit, residuals.*

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While presentations of instrumental variables methods are standard in econometric textbooks, including introductory ones, these books often do not present enough information to allow the practitioner conduct correct inference, even under *iid* settings. Not even a simple goodness of fit measure, such as R^2 is discussed. Econometric computer packages also suffer from this deficiency, in different degrees. In particular, none of the packages investigated[1] present an R^2 measure for instrumental variable models, although such measure has been studied almost ten years ago (Pesaran and Smith, 1994).

In textbooks, important exceptions are Davidson and McKinnon (1994) and, at a more introductory level, Johnston and DiNardo (1994). The later presents an *F*-test formula for models with *iid* residuals. These exceptions are small enough to motivate this note so to help practitioners. Notably, while most textbooks call into attention the need to calculate residual estimates correctly, Johnston and DiNardo's presentation, *e.g.*, make sure that there are different residual estimates to use, in addition to the one used to estimate error variances.

The purpose of this note is to present the different types of residuals from instrumental variables estimated models, some of their properties and collect from the literature a few results on model selection useful for practitioners.

TYPES OF RESIDUALS IN INSTRUMENTAL VARIABLE MODELS AND THEIR PROPERTIES

The basic instrumental variables setting in linear models is such that for the model $Y = X\beta + U$, where Y is an $n \times 1$ vector of response variables, X an $n \times k$ vector (including a constant term), β a $k \times 1$ vector of parameters and U an $n \times 1$ vector of error terms, $plim\ n^{-1}X'U \neq 0$, while consistency of least squares estimates $b=(X'X)^{-1}X'Y$ require that $plim\ n^{-1}X'U=0$. Suppose that there is an $l \times 1$ ($l \geq k$) matrix of instruments Z that fulfills the usual assumptions $plim\ n^{-1}Z'U=0$, $plim\ n^{-1}X'Z= Q_{XZ}$ a matrix of rank k and $plim\ n^{-1}Z'Z=Q_{ZZ}$, a positive definite matrix of rank l [2]. Under *iid* errors, it can be shown (*e.g.* Johnston and DiNardo, 1994) that the optimal method of moments estimator of β is $b_{IV}=(X'P_ZX)^{-1}X'P_ZY$, where $P_Z=Z(Z'Z)^{-1}Z'$, the well known projection matrix of Z. The estimator is also called two-stage least squares as the vector b can be obtained it two steps: first, regressing X on Z and saving the predicted values $x=P_ZX$, and second, using the predicted values in a regression of Y on x

Given an estimate of b_{IV}, most textbooks note that the "correct" estimates of the error vector U is given by $u = Y - Xb_{IV}$, while a rough and ready use of least square commands on the second stage of the estimation yield the "wrong" residuals $e=Y - xb_{IV}$.

The residual estimates e, also denoted prediction errors by Pesaran and Smith (1994) are called "wrong" estimates because the mean sum of squares does not yield a consistent estimate of the regression error variance $V(u_i)=\sigma^2$. Nevertheless, when comparing nested models, either by means of an F test or a

[1] PcGive 10, Shazam v.8, Limdep 7, Eviews 4 and SPSS v.10.
[2] As usual, $plim\ n^{-1}X'X= Q_{XX,}$ where Q_{XX} is a matrix of full rank k.

goodness of fit measure such as R^2, the "wrong" residuals should be used, for only they are orthogonal to the matrix of regressors X (and x of course).

These results can be better understood by writing

$$u = Y - Xb_{IV} = Y - X((X'P_ZX)^{-1}X'P_ZY)) = X\beta + U - X(X'P_ZX)^{-1}X'P_Z(X\beta + U)$$
$$= X\beta + U - X\beta - X(X'P_ZX)^{-1}X'P_Z U = U - X(X'P_ZX)^{-1}X'P_Z U$$
$$= [I - X(X'P_ZX)^{-1}X'P_Z]U;$$

where I is an identity matrix of length n; and

$$e = Y - xb_{IV} = Y - P_ZX((X'P_ZX)^{-1}X'P_ZY) = X\beta + U - P_ZX(X'P_ZX)^{-1}X'P_Z(X\beta + U)$$
$$= X\beta + U - P_ZX\beta - P_ZX(X'P_ZX)^{-1}X'P_Z U$$
$$= [I - P_Z] X\beta + [I - P_ZX(X'P_ZX)^{-1}X'P_Z]U = M_Z X\beta + M_xU.$$

Notice that the residual vector u is not exactly a transformation of error term by the "residual maker" matrix of x. If one recalls, in least squares models, the residual estimate $v = Y - Xb_{OLS}$, $b_{OLS} = (X'X)^{-1}X'Y$, can be written as $v = M_XU$, where $M_X=X(X'X)^{-1}X'$. On the other hand, the residual vector e has two components, one similar to the OLS results, given the second stage regressors, and another that presents the residual from the projection of the conditional mean of Y (*i.e.,* $X\beta$) on Z.[3]

In terms of mean squares, it can be seen that

$$u'u= ([I - X(X'P_ZX)^{-1}X'P_Z]U)'([I - X(X'P_ZX)^{-1}X'P_Z]U)$$
$$= U'U - U'P_ZX(X'P_ZX)^{-1}X'U - U'X(X'P_ZX)^{-1}X'P_ZU$$
$$+ U'P_ZX(X'P_ZX)^{-1}X'X(X'P_ZX)^{-1}X'P_Z U,$$

and the probability limit in mean is

$$plim\ n^{-1} u'u = plim\ n^{-1}U'U - 0 - 0 + 0 = \sigma^2.$$

The result follows from the assumptions of asymptotic ortogonality between U and Z.

For the "wrong" residuals one has,

$$e'e = (M_ZX\beta + M_xU)'(M_ZX\beta + M_xU)$$
$$= \beta'X'M_ZX\beta + U'M_x M_ZX\beta + \beta'X'M_Z M_xU + U'M_xU,$$

and the probability limit in mean is[4]

$$plim\ n^{-1} e'e = \beta'D\beta + 2\ plim\ n^{-1} U'X\beta + 0 + \sigma^2 + 0,$$

where $D = Q_{XX} - Q_{XZ}Q_{ZZ}^{-1}Q_{ZX}$, where D is also the estimate of the variance-covariance matrix of first stage residuals. The matrix will have many zero entries as many explanatory are also exogenous and the fist stage estimates will have a perfect fit. For a simple case of only one explanatory endogenous variable in X, $\beta'D\beta$ will reduce to the square of the respective β coefficient times the variance of the first stage residual. The second term on the right hand side does not vanish asymptotically, as it depends on the degree on endogeneity of X, that is the asymptotic correlation between the error term U and the explanatory variables X. Hence $n^{-1}e'e$ does not provide an asymptotically valid estimate of the error variance.

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[3] Startz (1983) presents some of these results with another notation.

[4] We use the fact that $M_ZM_X=M_Z$.

On the other hand, while u is not orthogonal to X, e is indeed orthogonal to x so it can be used to construct a sum of squares decomposition for IV models similar to ordinary least squares when the regressors are exogenous. That is, note that

$$Y'Y = (xb_{IV} + e)'\,(xb_{IV} + e) = b_{IV}'\,x'x'b_{IV} + 2\,b_{IV}'\,x'e + e'e,$$

but

$$\begin{aligned} x'e &= x'(M_Z X\beta + M_x U) = (P_Z X)'(M_Z X\beta + M_x U) \\ &= X'P_Z X\beta - X'P_Z P_Z X\beta + X'P_Z U - X'P_Z P_Z X(X'P_Z X)^{-1}X'P_Z U \\ &= X'P_Z X\beta - X'P_Z X\beta + X'P_Z U - X'P_Z U = 0. \end{aligned}$$

So that

$$Y'Y = b_{IV}'\,x'x'b_{IV} + e'e.$$

This can motivate a "generalized R^2" measure for IV models

$$GR^2 = 1 - e'e/\,Y'Y,$$

or, if X and Z include a column of ones and denoting $S_{YY}=(Y-\bar{y})'(Y-\bar{y})$, $\bar{y}=\Sigma y_i/n$

$$GR^2 = 1 - e'e/\,S_{YY}.$$

This generalized R^2 measure has been proposed by Pesaran and Smith (1994), where the asymptotic properties were investigated. The authors showed that this GR^2 measure has similar properties as the R^2 measure in ordinary least squares estimated linear models with X strictly exogenous. For example, $1 \geq GR^2 \geq 0$.

The results on this goodness of fit measure are important, as it is common to find statements that no such measure exist, even after more than five years the paper has been published. As noted in the introduction all those software either do not report a goodness of fit statistic for IV models or warn that the one present may not have desirable properties (as they are calculated using the residuals u).

Note that GR^2 measures goodness of fit of the model as a whole as first stage residual variances appear in the probability limit of $e'e$. Thus an improvement in GR^2 between two nested models may be due to better instruments *or* better regressors. This may be a reason for the "unpopularity" of this measure.

In addition, consider two models, one as above and the other with the same set of instruments but some coefficients, say, the last m coefficients in β restricted to zero. Denoting these models as unrestricted (*ur*) and restricted (*r*) respectively, it can be shown that in large samples (Pesaran and Smith, 1994, McKinnon , 1992)

$$S_{YY}\,(GR^2_{ur} - GR^2_{r})/(n^{-1}u'u) \sim \chi_m$$

or in the more familiar F-statistic form (Johnston and DiNardo, p.158)

$$S_{YY}\,[(GR^2_{ur} - GR^2_{r})/m]/[(u'u)/(n-k)] \quad \sim F_{m,\,n-k}.$$

The need to multiply by S_{YY} stems from the fact that $u'u$ does not appear in the sum of squares decomposition and that $n^{-1}e'e$ does not provide an asymptotically valid regression error variance estimate, as seen above.

FINAL COMMENTS

In this note we collected and made explicit a few results in the literature to clarify the differences between the possible residual estimates that can be obtained when estimating models by instrumental variables or two stage least squares. In addition we called into attention that there is a goodness of fit measure in the form of a generalized R^2 statistic available to the practitioners and that they are very simple to calculate. We hope that they begin to be used routinely and included in econometric software.

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Appendix: A few further results

For the sake of completeness we present a few further results on the orthogonality between the explanatory variables and the residuals considered in the model.

$$\begin{aligned} X'e &= X'(M_Z X\beta + M_x U) = X'(M_Z X\beta + M_x U) \\ &= X'X\beta - X'P_Z X\beta + X'U - X'P_Z X(X'P_Z X)^{-1} X'P_Z U \\ &= X'X\beta - X'P_Z X\beta + X'U - X'P_Z U = X'M_Z Y. \end{aligned}$$

$$\begin{aligned} x'u &= x'[I - X(X'P_Z X)^{-1} X'P_Z\,]U = X'P_Z U - X'P_Z X(X'P_Z X)^{-1} X'P_Z\,U \\ &= X'P_Z U - X'P_Z U = 0. \end{aligned}$$

$$\begin{aligned} X'u &= X'[I - X(X'P_Z X)^{-1} X'P_Z\,]U = X'U - X'P_Z X(X'P_Z X)^{-1} X'P_Z\,U \\ &= X'U - X'P_Z U = X'M_Z U. \end{aligned}$$

Note that although u is orthogonal to x (as e is to X), it does not hold that $Y=x'b+u$, so an R^2 measure cannot be constructed from this result.