Analysis and valuation of European-style and American-style exchange and spread options: the Brazilian case

Abstract: The aim of this work is to analyze both European-style and American-style exchange and spread options. With respect to the analysis of American-style options, we developed a model that adopted the Least-Squares Monte Carlo approach by Longstaff and Schwartz (2001) to consider the n-dimensional case. The results were compared with those obtained from the Rubinstein (1994) and Brandimarte (2006) models, both of which were based on the pyramidal lattice methods. We found that our model, which was an adaptation of the LSMC model, can be extended to three or more underlying assets compared with the original model.

Keywords: Derivatives; Options; Numerical Methods.

JEL Classification: G; G1; G13.

Resumo: O objetivo deste trabalho é analisar opções (cambias e spread) Européias e Americanas. Com relação à análise das opções Americanas, foi desenvolvido um modelo que adotou a abordagem de Mínimos Quadrados de Monte Carlo (LSMC) por Longstaff e Schwartz (2001) para analisar o caso n-dimensional. Os resultados foram comparados com os obtidos a partir dos modelos de Rubinstein (1994) e Brandimarte (2006), ambos os quais foram baseados nos métodos de redes piramidais. Descobrimos que o nosso modelo, que foi uma adaptação do modelo LSMC, pode ser estendido para três ou mais ativos subjacentes em comparação com o modelo original.

Palavras-chave: Derivativos; Opções; Métodos numéricos.

Classificação JEL: G; G1; G13

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1.Introduction

N-dimensional options are options with results that are the function of two or more underlying asset prices. In addition to exchange options and spread options, which are the objects of this study, the main examples of this derivative type include options of the minimum/maximum price of a set of assets, options on the ratio or the product of two or more asset prices and options on a portfolio of assets.

Margrabe (1978) was the first to obtain an analytical solution for pricing European-style exchange options; he demonstrated that an early exercise of these exchange options is unjustifiable if the underlying assets do not predict the distribution of dividends. Therefore, in the absence of dividends, the results found for European-style options are also valid for American-style call and put options. Additionally, Margrabe derived the parity relationships between American-style put and call options for exchange options.

Although exchange options may be seen as call options on asset 1 (with an exercise price equal to the price of asset 2) or put options on asset 2 (with an exercise price equal to the price of asset 1), Margrabe's (1978) aforementioned finding on exchange options does not conflict with that presented by Merton (1973), who demonstrated that an American-style vanilla call option on assets with no dividends should never be exercised before the expiration date. However, the same could not be said for put options. Merton's (1973) and Margrabe's (1978) conclusions differ in that the exercise price of vanilla options is fixed, but the exercise price of exchange options varies with time because exchange options are represented by the prices of other underlying assets.

Stulz (1982) developed analytical formulas to evaluate the pricing of European-style call and put options on the maximum and minimum prices of two assets. He showed that a call option on the minimum price of two assets can be evaluated by applying Margrabe's (1978) formula to exchange options while using a null exercise price. In other words, in the payoff calculations, the minimum price of underlying assets is not subtracted from the exercise price, which represents a fixed amount. Slutz (1982) also demonstrated that the price of a European-style call option on the minimum between two assets increases as the correlation between them gets close to 1. If the correlation coefficient between the assets increases, then there will also be an increase in the probability that the prices of asset 1 and asset 2 at maturity will be above the exercise price; as a result, the expected option payoff value is higher.

Johnson (1987) extended Slutz's (1982) findings by considering the pricing of call options on the minimum and maximum of multiple assets. In addition, Kirk (1995) developed an analytical solution for the pricing of European-style spread options. Notably, as will be seen in the next section in this paper, an exchange option is in fact a spread option with a null exercise price.

Cox et al. (1985) and Longstaff (1990) presented analytical solutions for several options on interest rates. Analogous to the exchange options studied by Margrabe (1978), these solutions included the option of exchanging one interest rate for another. In this context, Fu (1996) studied the problem of pricing European-style options with payoff represented by the difference between the same interest rates measured at different vertexes (yield-curve spread options). Fu's research supported the importance of using two-factor models to consider the imperfect movements of interest rates.

Berrahoui (2004) developed two methods to evaluate spread options and digital options with adjustment capacity as a function of the volatility curve behavior, i.e., the volatility smile. Berrahoui argues that the option price on the spread is sensitive to the volatility curve format of the underlying assets and that the highest effect occurs on digital options. Berrahoui's first method changes the exercise price of the option into a function of the volatility of each underlying asset such that representing only a partial part of the volatility smile. The second method takes the total volatility curve into account and involves more complex numerical integration calculations.

Although several authors have developed analytical solutions for the pricing of n-dimensional options, these solutions are only applicable to some specific cases and do not evaluate such possibilities as the early exercise of American-style options. Thus, in addition to the development of analytical solutions (or closed solutions) for problems with derivatives that depend on more than one underlying asset, there is a critical need for improvement in the numerical methods used to solve several related problems. Moreover, even if an analytical solution is possible, researchers should also consider developing alternative solutions that, in addition to producing the same results as those presented by an analytical solution, also perform additional analyses, such as calculating the probability of exercising options through the Monte Carlo Simulation method or calculating sensitivity measurements (Greek letters) to prepare hedge strategies.

Numerical methods can be divided into three basic categories: methods for solving partial differential equations, especially the finite differences methods first introduced to the financial field by Brennan and Schwartz (1977); the Monte Carlo Simulation method, which was first applied by Boyle (1977); and lattice methods, which were initially proposed by Cox et al. (1979).

The Monte Carlo Simulation method has been frequently used to evaluate several types of derivatives. In the financial literature, the Monte Carlo Simulation has been widely used to (1) calculate options in market and credit risk measurement, (2) calculate the Value at Risk (VaR), (3) evaluate projects through Real Options, and so on. With respect to the evaluation of options, some models ignore the belief that simulation methods are only applicable to the evaluation of European-style options. According to the traditional view, these methods do not have the flexibility needed to evaluate American-style options. One of the models that ignores this view is the Least-Squares Monte Carlo method created by Longstaff and Schwartz (2001).

Based on Monte Carlo Simulations, Villani (2007) developed models to price the three types of n-dimensional options: European-style exchange options on underlying assets that contemplate the distribution of dividends; European-style exchange options in which the underlying asset is another exchange option (compound exchange); and Pseudo American-style exchange options, also known as Bermuda options, that represent options with more than one exercise date. In his models, Villani (2007) reduced to two-dimensionality the exchange-options evaluation problem by simulating the values of the ratio between the prices of underlying assets. Villani's results were comparable to those described by Margrabe (1978), Carr (1988), and Armada et al. (2007).

Venkatramanan (2005), Boyle (1988), Boyle et al. (1989), Kamrad and Ritchken (1991), Brandimarte (2006) and Rubinstein (1994) used lattice approaches to address n-dimensional options. These models present good flexibility, which allows for an intuitive interpretation of the problem. Venkatramanan (2005) presented a three-dimensional lattice model with n-dimensional interpolation to evaluate European-style and American-style options on two or more assets. Similar to Berrahoui (2004), Venkatramanan's model allows adjustment capacity as a function of the volatility curve behavior, i.e., the volatility smile.

Boyle (1988) developed a trinomial model to evaluate American-style put options on the minimum price between two underlying assets. For these two underlying assets, the probability that their prices change are obtained by comparing the first two discontinued points of the distribution with the first two points of the joint lognormal distribution of the returns. Thus, Boyle assumed that the prices of the underlying assets' probability distribution was bivariate lognormal. Boyle's model considers five alternative price trajectories as the approximations of the bivariate joint lognormal process for the prices of the two underlying assets.

Because Boyle's model (1988) can generate negative probabilities for the price changes of the underlying assets, Boyle et al. (1989) considered an alternative model to evaluate European-style put options on multiple assets. In the new model, four price movements are used to approximate the stochastic process for the two asset prices' logarithmic returns. Boyle calculated the moment generating function of the approximated distribution by using the moment generating function of the normal distribution. As a result, for each time change, the problem is the solution of a system composed of four equations and four variables. Boyle then used the only probabilistic expression for the multinomial distribution to evaluate the derivative being studied. Although Boyle et al. (1989) evaluated the accuracy of the model by applying it to a European-style option with three underlying assets, this method can also easily predict if a person will exercise American-style options before their expiration.

Kamrad and Ritchken (1991) developed an alternative and more generalized methodology to evaluate derivatives on one or more underlying assets. Similar to Boyle et al. (1989), Kamrad and Richken's model also approximates the stochastic process for the asset prices' logarithmic return by using a multinomial tree. However, unlike Boyle et al. (1989), Kamrad and Ritchken's (1991) model allows horizontal price movements and demonstrated that the binomial model created by Cox et al. (1979) is a particular case of yours model when applied to only one state variable. However, Boyle et al.'s (1989) model is a special case of Kamrad and Ritchken's (1991) model when applied to two state variables.

Brandimarte (2006) and Rubinstein (1994) also use the lattice approach to evaluate the options on two correlated assets. Their models are based on a three-dimensional tree in which the discretions of two correlated stochastic processes are represented as binomial pyramids. The Rubinstein model considers the correlation between the underlying assets by examining the probabilities for their prices.

The present paper evaluates European-style and American-style exchange and spread options by applying some of the aforementioned methods. This paper is divided into five sections. Section 1 consists of the introduction, and item 2 describes the general aspects of the n-dimensional options as well as their characteristics and particularities. Item 3 specifically addresses exchange options and European-style spread options. The model proposed here to evaluate these derivatives is based on the Monte Carlo Simulation method, which conducts a joint simulation of the underlying assets, which is necessary to verify the correlation between these assets. After presentation of the model, it is used to evaluate both the exchange options and the European-style spread options. To judge the consistency of the results, Margrabe's formula was applied to the exchange options and Kirk's formula was applied to the spread options, to compare the premiums of the options generated[1].

After evaluating the European-style option, in item 4, the exchange option and American-style spread options are addressed by considering three specific models. The first and second models apply Rubinstein's (1994) and Brandimarte's (2006) lattice methods and binomial pyramids, respectively. The third model addresses American-style options and applies Longstaff and Schwartz's (2001) Least-Squares Monte Carlo (LSMC) approach. This model was originally developed to evaluate American-style vanilla options, but we adapted it to analyze n-dimensional options. Thus, the next stage applies the three models – lattices by Rubinstein (1994) and by Brandimarte (2006) and LSMC by Longstaff and Schwartz (2001) – to consider some American-style exchange options and spread options. In addition, the results of each methodology are compared. Finally, item 7 is dedicated to final commentary and other considerations.

[1] Formulas proposed by Kirk and Margrabe are presented in appendix 1.

2. Exchange, spread and other n-dimensional options

The main feature of n-dimensional options is the payoff, which depends on the price of two or more assets[2]. In this context, S_i is considered to be the price of asset i, and the following examples of n-dimensional options are presented:

- Exchange options → Payoff is based on the price difference between two assets.

Call option payoff : $\max(S_2 - S_1, 0)$ (1)

Put option payoff : $\max(S_1 \quad S_2, 0)$ (2)

- Option on the maximum price for two or more assets → As the name indicates, payoff is based on the maximum price between two or more assets. A direct application of this type of option consists of a hedge of assets portfolios with negative correlations.

Call option payoff for two underlying assets: $\max[\max(S_1, S_2) \quad X, 0]$ (3)

Put option payoff for two underlying assets: $\max[X \quad \max(S_2, S_1), 0]$ (4)

- Option on the minimum price for two or more assets → As the name indicates, payoff is based on the maximum price between two or more assets. The direct application of this type of option is the same as the previously described application: a hedge of asset portfolios with negative correlations.

Call option payoff for two underlying assets: $mín[mín(S_1, S_2) - X, 0]$ (5)

Put option payoff for two underlying assets: $mín[X - mín(S_2, S_1), 0]$ (6)

- Spread options → As with exchange options, payoff is based on the price difference between two assets. However, this calculation also considers an exercise price. In fact, exchange options are essentially spread options with a null exercise price.

Call option payoff: $\max[(S_2 - S_1) - X, 0]$ (7)

Put option payoff: $\max[X - (S_2 - S_1), 0]$ (8)

- Option with double exercise price → Payoff is based on two exercise prices, each of which is dependent on a distinct underlying asset.

Call option payoff: $\max[(S_1 - K_1), (S_2 - K_2), 0]$ (9)

Put option payoff: $\max[(K_1 - S_1), (K_2 - S_2), 0]$ (10)

- Option on the ratio between prices of underlying assets → As the name indicates, payoff is based on the relation between the prices of two assets.

Call option payoff: $\max\left(S_2/S_1 - X, 0\right)$ (11)

Put option payoff: $\max\left(X - S_2/S_1, 0\right)$ (12)

- Option on the product of prices of the underlying assets → As the name indicates, payoff is based on the product of the prices of two assets.

Call option payoff: $\max\left(S_2 \times S_1 - X, 0\right)$ (13)

Put option payoff: $\max\left(X - S_2 \times S_1, 0\right)$ (14)

[2] Some sub-classifications of n-dimensional options are the Rainbow Options, which usually establish the price of three or more assets, or "Basket Options," which refer to a basket of assets.

- Options on portfolios → Payoff is based on the value of a portfolio with specific percentages allocated to two or more underlying assets.

  Call option payoff: $\max[(n_1 S_1 + \ldots + n_m S_m) - X, 0]$, where $n_i > 0$ (15)

  Put option payoff: $\max[X - (n_1 S_1 + \ldots + n_m S_m), 0]$, where $n_i > 0$ (16)

Exchange and spread options can be negotiated on a wide variety of assets, including stocks, interest rates, commodities and futures contracts. Therefore, these options have the potential to represent hedging and investment opportunities for different players in the market.

For instance, in the fixed-income market, the spread option payoff may be defined as the difference between two or more interest rates. This definition was used to determine the profitability difference in Mortgage Over U.S. Treasury Options (*MOTTO*), which were offered by Goldman Sachs. Additionally, the spread option payoff may be defined as the same interest rate observed at different vertexes, as in the case of Slope-of-the-Yield-Curve Options, which were also offered by Goldman Sachs. In addition, Spread-Lock Options represent options on interest rates swaps that provide to call option holder with the right to buy an interest rate swap. These interest rate swaps represent the possibility of receiving an amount corrected by the difference between a fixed rate and a floating rate. However, provide to put option holder with the right to sell an interest rate swap.

In the commodities market, both exchange options and spread options may adopt the difference between several types of prices, such as location spreads options, calendar spreads options, processing spreads options, and quality spreads options, as reference point. In the energy market, most actors deal with so-called crack spread options and spark spread options. On the one hand, crack spread options represent the simultaneous purchase or sale of crude oil and a refined oil by-product. In this case, the price difference between these underlying assets represents the value added through oil refinement. On the other hand, spark spread options represent the cost proxy incurred by converting some specific fuel (usually natural gas) into electrical power. In practice, spark spread options are based on the difference between the price of the fuel used to generate another type of energy and the price of this fuel after it reaches the final consumption form.

Although these derivatives are usually negotiated in the over-the-counter market, they can also be negotiated in the stock exchanges. For example, to help energy market actors better manage price variation risks, the New York Mercantile Exchange (NYMEX) introduced the crack spread options in 1994, which offered American spread options based on the price differences between heating oil and crude oil and the price differences between RBOBO (Reformulated Gasoline Blendstock for Oxygen Blending) and crude oil.

In addition, since 2002, the NYMEX has offered calendar spread options based on the price differences between futures contracts with distinct maturities. These futures contracts, in turn, may adopt the prices of several products, such as heating oil, electrical power, crude oil, natural gas and gasoline, as a reference point. The buyer of a put option of this kind assumes a short position in the futures market for a shorter maturity and a long position in the futures market for a longer maturity. Thus, the buyer can obtain a hedge against the possibility that the loading costs of the reference asset in the futures contracts become higher. The buyer of a call option assumes the opposite position because a short position in the futures market denotes longer maturity and a long position in the futures market denotes shorter maturity. In this context, calendar spread options represent a more practical and usually cheaper hedge alternative compared with futures contracts with different maturity dates.

Spread Options are also present in the agricultural market, as demonstrated by the so-called Crush Spread Options offered in the Chicago Board of Trade (CBOT). In addition to soybean futures contracts, the other reference asset in this market is a soybean oil or soybean grain futures contracts. In turn, the value of the spread option offered by the CBOT depends on the difference between the prices of the grain or oil extracted from a bushel (eight gallons). Thus, is a Processing Spread Options, where the crude product is soybean and its by-products are soybean oil and grain.

Although exchange options and spread options are not yet offered in Brazil, they could be implemented given the differences between the prices of a large variety of assets, such as interest rates, commodities, futures contracts, swaps, common stocks and preferred capital stocks.

Options based on two or more distinct stocks should produce the same effects as the simultaneous purchase and sale of these stocks in the stock market. In this case, however, the need to possess this asset or the obligation to lease it and the need to purchase it or to sell it in the stock market would no longer exist. As a result, a market player could take advantage of the price movements between the reference assets with lower costs and those with higher financial leverage. Azevedo and Barbachan (2005) compared the results obtained from purchasing European-style exchange options with those obtained from simultaneously selling the underlying reference assets in the option, demonstrating that if the market moves contrary to the expected direction, then the resulting financial losses can be minimized through the use of exchange options.

Another application of exchange options and spread options in the Brazilian market would be the use of spread-lock options, which could refer to several swaps negotiated in the BM&FBOVESPA. In addition, spread-lock options could refer to the implementation of calendar spread options based on the price differences between futures contracts with distinct maturities, such as coffee mini futures, cattle, dollar, and IBOVESPA or any other standard futures contract negotiated in the stock exchange. Thus, the negotiation of these derivatives could represent a more practical and cheaper hedge against the possibility that the loading costs of the reference asset in futures contracts undergo unfavorable changes.

3. Evaluation of European-style exchange and spread options through the Monte Carlo Simulation

3.1. Description of the model

The evaluation of vanilla European-style options through the Monte Carlo simulation consists of the following three stages:

1. Simulation of the underlying asset values in the option's maturity date (risk-neutral world).
2. Settlement of the derivative payoff.
3. Pricing of the option through the current average value of payoffs at the risk-free interest rate (risk-neutral world).

Because the price of the underlying asset follows the Geometric Brownian Motion, the adequate discrete evolution model of the Neperian logarithm in relation to the price of the underlying asset is the following:

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2} - q\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}} \tag{17}$$

In a risk-neutral world, the expected investment return rate of the underlying asset is already considered in its price. As a result, knowledge of the expected investment return rate is unnecessary. Thus, before beginning the simulations, μ is replaced by the risk-free interest rate in equation (17). From the simulation of values for ε , which follows a standardized normal distribution, the potential trajectories of the underlying asset's value with respect to time will be defined. As the number of trajectories increases, we can begin to see a distribution of prices for the underlying asset at the option's maturity date. For the trajectories that are sufficiently large to obtain convergence for the results, the current payoff value of each trajectory will determine the premium of the option being analyzed.

However, to evaluate European-style exchange and spread options, we must consider the correlation between the reference assets. Thus, a joint simulation of the price trajectories of these assets should be performed.

As previously seen, the payoff for spread call options is the following:

Spread options: $\max[(S_2 - S_1) - X, 0]$ (18)

Exchange options are in fact spread options with a null exercise price. The payoff of exchange call options is[3] the following:

Exchange options: $\max(S_2 - S_1, 0)$ (19)

If it is assumed that the continuous dividends rates q_1 and q_2 are distributed through assets S_1 and S_2, respectively, then the price trajectories of each reference asset may be represented by the following stochastic process:

$$dS_1 = (r - q_1)S_1 dt + \sigma_1 S_1 dZ_1 \tag{20}$$
$$dS_2 = (r - q_2)S_2 dt + \sigma_2 S_2 dZ_2 \tag{21}$$

where dZ1 and dZ2 are Wiener processes with instantaneous correlation $\rho_{1,2}$:

$$dZ_1 dZ_2 = \rho_{1,2} dt \tag{22}$$

Thus, to apply the Monte Carlo Simulation Method, a conjoint simulation of the stochastic processes should be considered, as presented in equations (20) and (21). To do so, from the covariance matrix corresponding to two variables with standardized normal distribution and correlation $\rho_{1,2}$, the Cholesky decomposition should be used to find the only upper triangular matrix L such that $\sum = L'L$:

[3] In both cases (spread and exchange options), the payoff of put options are obtained from inverting the signs of S1, S2 and X presented in the call options payoff.

$$\sum = \begin{bmatrix} 1 & \rho_{1,2} \\ \rho_{1,2} & 1 \end{bmatrix} \quad (23)$$

By multiplying the matrix L by its transpose, the matrix presented in equation (23) is obtained. Thus, the matrix L is the Cholesky factor for $\sum$:

$$L = \begin{bmatrix} 1 & \rho_{1,2} \\ 0 & \sqrt{1-\rho_{1,2}^2} \end{bmatrix} \quad (24)$$

If a vector with n independent random variables with standard normal distribution is considered, the following is obtained:

$$Z = \left(Z_1, Z_2, \ldots Z_n\right)' \quad (25)$$

The correlated random variables vector $\varepsilon = \left(\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n\right)'$ is then defined as the following:

$$\varepsilon = L'Z \quad (26)$$

Thus, to simulate two correlated random variables (n=2), the following is executed:

$$\varepsilon = \left(\varepsilon_1, \varepsilon_2\right)' \text{ and } Z = \left(Z_1, Z_2\right)' \quad (27)$$

Therefore, to simulate two correlated Wiener processes and to determine the price trajectory of asset prices S_1 and S_2, two independent variables, Z_1 and Z_2, should be generated with standard normal distribution. Thus, the trajectory S_1 is obtained based on ε_1, and trajectory S_2 is obtained based on ε_2, as shown below:

$$\varepsilon_1 = Z_1 \quad (28)$$

$$\varepsilon_2 = \rho_{1,2} Z_1 + \sqrt{1-\rho_{1,2}^2}\, Z_2 \quad (29)$$

The only distinction between the methodology used for the evaluation of exchange options and that used for the evaluation of spread options is the criterion used to define the payoff on the maturity date, as shown in equations (18) and (19).

3.2. Application of the model

To analyze the model based on Monte Carlo Simulations, it is used to estimate the prices of both European-style exchange and spread options based on specific pairs of stocks. The options with common references and preferred capital stocks of the same company will be analyzed, as well as options with references in the preferred capital stocks of two distinct companies from the same sector. The model's estimated premiums are then compared with those obtained by applying the formulas of Margrabe (1978) and Kirk (1995) to the exchange options and spread options, respectively. The results allow us to evaluate the model's accuracy (Appendix).

As previously mentioned, these options produced similar effects to the purchase and sale of the underlying stocks in the Stock Market, but they produced potentially lower costs because the physical exchange and the need to either possess or lease this asset no longer exist, because the option payoff could be paid through the difference. Moreover, this instrument provides actors with a higher leverage, which facilitates the performance of both hedges and bets in relatively favorable movements.

This study will analyze the exchange and spread options between the common stocks and the preferred capital stocks of two companies, Petrobras (codes PETR3 and PETR4, respectively) and Vale (codes VALE3 and VALE5, respectively). The exchange options and the spread options between the preferred capital stocks of the companies Petrobras and Ipiranga (codes PETR4 and PTIP4, respectively) will also be analyzed, between the preferred capital stocks of the companies Bradesco and Itaú (codes BBDC4 and ITAU4, respectively) and between the preferred capital stocks of the companies Usiminas and Acesita (codes USIM5 and ACES4, respectively). Table 1 presents the data corresponding to the stocks above.

Table 1. Data corresponding to stocks analyzed.

Sector	Company	Stock code	Type of share	Closing price (July 25, 2007)	Annualized standard-deviation	Standard-deviation (0.1 day)
Oil	Petrobras	PETR3	common	63.40	2.156%	0.6817 %
		PETR4	preferred	54.74	2.022%	0.6394%
Oil	Ipiranga	PTIP4	preferred	25.75	2.971%	0.9395 %
Banking	Bradesco	BBDC3	common	52.45	2.478%	0.7836 %
		BBDC4	preferred	51.48	2.056%	0.6501%
Banking	Itaú	ITAU4	preferred	90.00	2.119%	0.6702 %
Mining	Vale	VALE3	common	95.00	2.981%	0.9427 %
		VALE5	preferred	81.15	2.830%	0.8949%
Metallurgy	Usiminas	USIM5	preferred	115.00	2.964%	0.9374 %
Metallurgy	Acesita	ACES4	preferred	73.10	1.980%	0.6262 %

Source: Own elaboration

Exchange option payoffs are defined by the difference of prices between underlying assets. Therefore, the payoffs of each exchange option are the following: max(PETR3 – PETR4, 0), max(BBDC3 – BBDC4, 0), max(VALE3 – VALE5, 0), max(PETR4 – PTIP4, 0), max(ITAU4 – BBDC4, 0), and max(USIM5 – ACES4, 0).

In this work, Spread options payoffs consider the subtraction of the price differences between the underlying stocks verified on July 25th 2007. Thus, based on the closing prices shown in Table 1, the payoffs of each spread option are the following: max(PETR3

– PETR4 – 8.66, 0), max(BBDC3 – BBDC4 – 0.97, 0), max(VALE3 – VALE5 – 13.85, 0), max(PETR4 – PTIP4 – 28.99, 0), max(ITAU4 – BBDC4 – 38.52, 0) and max(USIM5 – ACES4 – 41.9, 0).

In addition to the information described above, the other input parameters used in the model's application are the following:

- The number of working days until the maturity date of the options is 66 days. The time step performed for both models was 0.1 day.
- The risk-free interest rate was 0.901476% on a monthly basis, which is equivalent to the CDI interest rate (Interbank Deposit Certificate) at July 2007 or 0.002992% in the period of 0.1 day (continuous capitalization).
- The volatility was estimated based on the standard deviations of the historical logarithmic returns observed for each underlying asset between June 15th, 2006 and June 15th, 2007. Table 1 shows the standard deviations for one year and for a period of 0.1 day, which was estimated for each reference stock.
- The correlations of the underlying stocks logarithmic returns between June 15th, 2006 and June 15th, 2007 are the following: PETR3 and PETR4 (0.99), BBDC3 and BBDC4 (0.98), VALE3 and VALE5 (0.99), PETR4 and PTIP4 (0.84), ITAU4 and BBDC4 (0.97), USIM5 and ACES4 (0.78).
- The number of experiments and Monte Carlo Simulations performed in each calculation are 20 experiments and 10.000 simulations, respectively.

Table 2 presents a comparison of the results generated by the Monte Carlo Simulation model and the Margrabe (1978) model for each of the exchange options tested in this experiment. In addition, the table presents the correlation between the reference assets in each option. We observed a good adherence between the results presented by both methods.

Table 2. Evaluation of European-style Exchange Options.

	PETR3 and PETR4	BBDC3 and BBDC4	VALE3 and VALE5	PETR4 and PTIP4	ITAU4 and BBDC4	USIM5 and ACES4
Margrabe (M)	8.660	1.597	13.850	28.990	38.520	41.905
Monte Carlo (MC)	8.661	1.596	13.848	28.988	38.520	41.894
(M-MC)/M	-0.01%	0.06%	0.01%	0.01%	0.00%	0.03%

Source: Own elaboration

Table 3 presents a comparison of the results generated by the Monte Carlo Simulation model and the Kirk (1995) model for each of the spread options studied in this experiment. A good adherence between the results presented by both methods was observed. Although the Monte Carlo Simulation model tended to generate higher results than those generated by the Kirk (1995) model, the highest discrepancy was only 0.94%, which belonged to the evaluation of the spread option on stocks ITAU4 and BBDC4.

Table 3. Evaluation of European-style Spread Options.

	PETR3 and PETR4	BBDC3 and BBDC4	VALE3 and VALE5	PETR4 and PTIP4	ITAU4 and BBDC4	USIM5 and ACES4
Kirk (K)	1.088	1.098	2.205	3.205	3.303	8.269
Monte Carlo (MC)	1.091	1.098	2.210	3.225	3.334	8.309
(K-MC)/K	-0.28%	0.00%	-0.23%	-0.62%	-0.94%	-0.48%

Source: Own elaboration

As seen above, exchange options are in fact spread options with a null exercise price. Table 4 presents a comparison of the results corresponding to the evaluation of the exchange options, as shown in Table 2. The results were generated by the Monte Carlo and Kirk (1995) models for the same spread options studied previously, but these results included an exercise price equal to 0.01 as well. Hence, a coherent evaluation of the spread options should generate values approximately equal to the estimated values for the exchange options. As Table 4 shows, this prediction is verified in all results generated by the Monte Carlo Simulation model. However, the Kirk (1995) model generated inconsistent results. As previously mentioned, the only difference between the Monte Carlo Simulation model used to evaluate the spread options and the model used to evaluate the exchange options is the payoff performed at the exercise date. Hence, the Kirk (1995) model's problems with evaluating the spread options may stem from its underestimation of the option value.

Table 4. Evaluation of European-style Exchange Options (EO) and Spread Options (SO) with exercise price close to zero.

	PETR3 and PETR4	BBDC3 and BBDC4	VALE3 and VALE5	PETR4 and PTIP4	ITAU4 and BBDC4	USIM5 and ACES4
Margrabe (EO)	8.660	1.597	13.850	28.990	38.520	41.905
Monte Carlo (EO)	8.661	1.596	13.848	28.988	38.520	41.894
Kirk (SO)	5.401	1.519	8.930	23.365	28.164	32.578
Monte Carlo (SO)	8.649	1.589	13.833	28.981	38.516	41.890

Source: Own elaboration

3.2.1. Sensitivity analysis

Some of the parameters of the models can be changed to verify the option premium's sensitivity to the changed variable. In this section is studied the relationship between the prices of European-style exchange and spread options to changes in the volatility of the underlying assets, in the correlation of the underlying stock prices and in the risk-free interest rates. The premium sensitivity of these exchange options can be analyzed in relation to changes in other parameters, such as maturity date. In addition the relation between spread options and change in the exercise price can be analyzed.

Based on the Monte Carlo Simulation model and the Margrabe (1978) formula, Table 5 analyzes the price behaviors of European-style exchange options for the common stocks and the preferred capital stocks of Petrobras (codes, PETR 3 and PETR4, respectively) in relation to the 30% reduction in the correlation between PETR3 and PETR4 (from 0.99 to 0.69), the 50% increase in the daily volatility of PETR3 (from 2.16% to 3.23%) and PETR4 (from 2.02% to 3.03%) and the 50% increase in the risk-free interest rate (from 0.9015% to 1.3522%).

Each model presented similar results. Based on these observations, the price of the option reacted positively to the reduction in the correlation and the increase in the volatility of the underlying assets. This behavior was expected, as both changes had increased the chances for higher differences between the prices of PETR3 and PETR4 at the maturity date; thus, their payoff value expectancy increased as well.

However, the change in the risk-free interest rate exerted little influence on the price of the option, although this change caused some reduction in the option's value. This result may seem unusual at first because the risk-free interest rate positively affects the premium of a vanilla call option[4]. The almost null effect of the interest rate on the exchange options is caused by the payoff characteristics of these options. In a vanilla call option, the payoff is defined as the difference between the price of the underlying asset and the exercise price, which is represented by a fixed amount.

On the other hand, the exchange option payoff is defined as the difference between the prices of the two underlying assets. Thus, in a risk-free world, the expected value of each underlying asset increases in proportion to the increase in both the interest rate and the difference between the asset prices. The increase in the expected value nullifies the negative effect on the option's price. This negative effect was caused by the increase in the discount rate used to calculate the present value of cash flow.

Table 5. Price sensitivity of the European-style Exchange Option (PETR3 and PETR4).

	Monte Carlo	Margrabe
Base Scenario	8.661	8.660
Reduction of 30% on the correlation	9.190	9.196
Increase of 50% on the volatility of PETR3	8.870	8.867
Increase of 50% on the volatility of PETR5	8.709	8.704
Increase of 50% on the risk-free increase rate	8.660	8.658

Source: Own elaboration

In addition to the analysis presented in Table 5 and based on Margrabe's (1978) formula, the values of the same exchange option were estimated while considering different volatility levels for PETR3 and PETR4. The results are presented in the first graphic of Fig. 1.

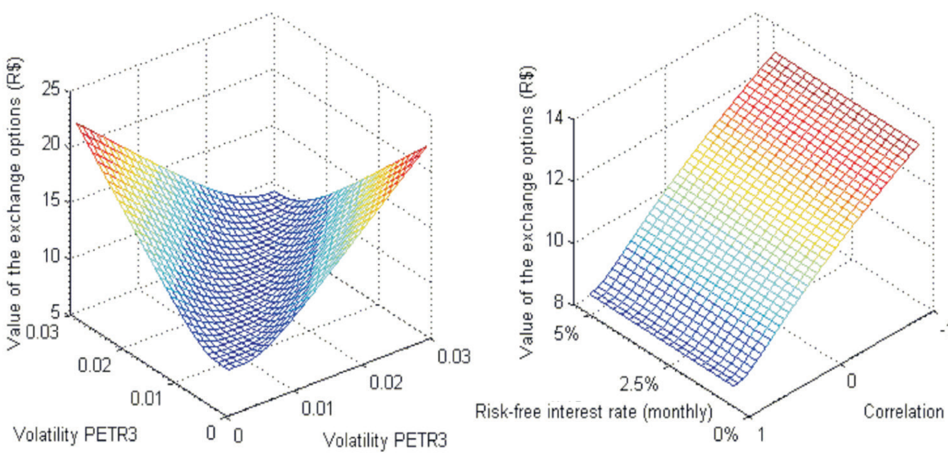
The value of the option increases as the volatility between the underlying assets becomes increasingly different from each other; the lowest values are observed when the volatilities of the underlying assets are the same. This observation is interesting. Because the correlation between PETR3 and PETR4 is high (0.99), when their volatilities are

[4] The higher the risk-free interest rate, the lower the current exercise price will be. Because the exercise price represents a possible expense for the vanilla call option holder, it will be in a better situation when the interest rate is higher.

similar, the probability of a divergence in their values on the maturity date is lower, and thus, the option's estimated value is reduced.

In relation to Figure 1, the analysis presented in the second graphic refers to the price behavior of the exchange options in relation to the changes in the coefficient of the correlation between PETR 3 and PETR4 at different interest rates. Again, Margrabe's (1978) formula is considered for this analysis. In relation to the changes in the correlation coefficient, the changes observed in the premiums are in agreement with the changes observed by the financial institution. That is, the premiums are inversely proportional to the correlation between the two reference stocks. However, in relation to the changes in the interest rates, the value of the exchange option presents no sensitivity at all, which agrees with the results presented in Table 5.

Fig. 1. Value sensitivity of Exchange Options (R$) in relation to the changes in the volatility of the underlying assets, the changes in the risk-free interest rates and the correlation between the underlying assets.



Source: Own elaboration

Based on the Monte Carlo Simulation model and Kirk's (1995) formula, Table 6 presents a value sensitivity analysis of the European-style spread options for PETR3 and PETR4. This analysis was conducted in relation to the same changes previously applied to the evaluation parameters of the same exchange options.

Each model also presents similar results. We observe that the option's price reacted positively to the reduction in the correlation and to the increase in the volatility of the underlying assets. In addition, the change in the correlation strongly influenced the price of the spread option in relation to the price of the exchange option, and the effect generated by the increased volatility of PETR3 was considerably higher than that produced by the increased volatility of PETR4. This behavior occurs as a function of the difference between the initial prices of the underlying assets and the subtraction of the exercise price during the payoff of the spread options.

With respect to the change in the interest rate, unlike exchange options, spread options presented strong sensitivity, as they reacted positively to the increases in the interest rate. As mentioned previously, in a risk-free world, the expected value of each underlying asset (and the difference between the prices as well) increases proportionally as the interest rate increases. In exchange options, this effect nullifies the negative effect on the option's price, which was caused by the increase in the discount rate.

However, the same cannot be said for spread options. In spread options, even if the differences between the prices of the underlying assets increases at the same proportion as those of an exchange option, this effect will still have a positive impact on the spread option's price because the exercise price becomes relatively smaller (the exercise price appears with a negative sign in the payoff).

Table 6 . Price sensitivity of the European-style Spread Option (PETR3 and PETR4).

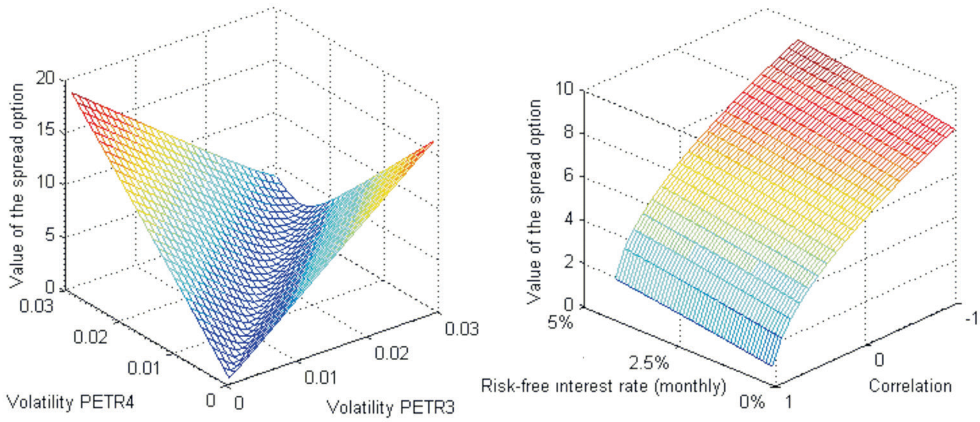
	Monte Carlo	Kirk
Base Scenario	1.091	1.088
Reduction of 30% on the correlation	3.307	3.306
Increase of 50% on the volatility of PETR3	3.199	3.195
Increase of 50% on the volatility of PETR5	1.280	1.269
Increase of 50% on the risk-free interest rate	1.130	1.125

Source: Own elaboration

In addition to the analysis that was presented in table 6 and based on Kirk's (1995) formula, the values of the same spread options were estimated while considering different volatility levels for PETR3 and PETR4. The results are presented in the first graphic of Fig. 2. An interesting fact is that there is not symmetry observed in the same graph referring to the exchange options (Fig. 1), which occurs due to the difference between the initial prices of these stocks and the subtraction of the spread options payoffs.

In relation to Figure 2, the analysis presented in the second graphic refers to the price behaviors of the spread options for each change in the correlation coefficient between PETR3 and PETR4 and in the risk-free interest rate. Again, Kirk's (1995) formula is applied to our analysis. As previously observed, premiums inversely proportional to the value of this parameter (i.e., change in the correlation coefficient) could be observed. However, unlike exchange options, spread options present sensitivity to changes in the interest rate, as explained previously.

Fig. 2. Value sensitivity of spread options (R$) in relation to changes in the volatility of underlying assets, changes in the risk-free interest rates and correlation between the underlying assets.



Source: Own elaboration

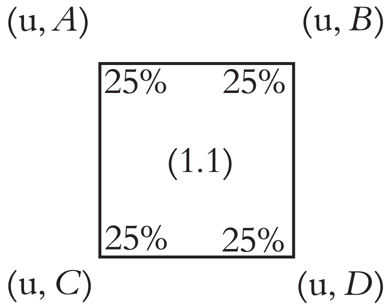
4. Analysis of American-style exchange and spread options through binomial pyramids and the least-squares Monte Carlo method

This item deals with the pricing of exchange options and American-style spread options by using binomial pyramids and the Least-Squares Monte Carlo Method. With respect to binomial pyramid-based models, the application of Rubinstein's (1994) model, where the correlation between underlying assets is incorporated based on the possible returns for each asset, will be specifically considered. In Brandimarte's (2006) model, the correlation between underlying assets is incorporated based on the probabilities for each price movement. Thus, after presenting the three models, they will be applied and the results of each will be compared.

4.1. Description of Rubinstein's pyramidal lattice model

The first step to defining Rubinstein's model is to standardize the initial prices of the underlying assets in the pair (1,1). Similar to the high or low movements of a single asset, Rubinstein's binomial pyramid also considers the initial movement of the first underlying asset (i.e., asset 1) to be represented by two specific returns, d or u. In addition, the probabilities of the returns d or u in Rubinstein's model are considered to be equal (i.e., they each assume a value of 50%). On the one hand, if the return of asset 1 is u, then the second asset (i.e., asset 2) presents two possible returns, A or B, with equal probabilities. On the other hand, if the return of asset 1 is d, then asset 2 presents two possible returns, *C* or *D*, with equal probabilities. The possible returns of asset 2 are selected such that AD = BC, which becomes crucial for there to be the crossroads of several trajectories of prices started differently. Fig. 3 represents the first price movement of both underlying assets and indicates that the initial pair (1,1) can move to (u,A), (u,B), (d,C) or (d,D), with equal probability for each movement.

Figure 3. First movement of prices.

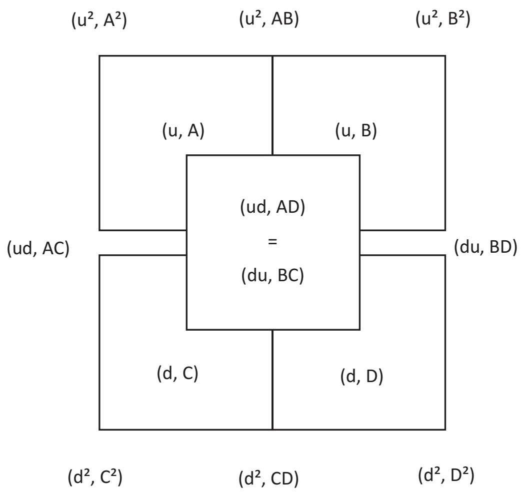


Source: Own elaboration

As previously mentioned, Rubinstein's (1994) model incorporates the correlation between underlying assets by using their possible returns. To contemplate a correlation coefficient different from zero, the performance of the first initial return for the first asset should modify the possible returns (or the return probabilities) of the second asset. In this context, a correlation different from zero should adopt $A \neq C$ and $B \neq D$ in the binomial pyramid. For example, if $A = C$ and $B = D$, then the first asset moves up or down, and the possible movements of the second asset and its respective occurrence probabilities are kept exactly equal, which indicates a null correlation between the assets. If asset 1 moves up, then asset 2 will move to A or B, with a 25% probability of performing each action. However, if asset 1 moves down, then asset 2 will move to C or D, with a 25% probability of performing each action. If $A = C$ and $B = D$, then the movements of A and B that have a 25% probability of occurring are equal to those of C and D.

Figure 4 depicts the second price movement of each underlying asset. For example, given the first return (u,A), the second return may be (u,A), (u,B), (d,C) or (d,D). Hence, the total return of the first two price movements for both underlying assets will be (u^2, A^2), (u^2, AB), (ud, AC) or (ud, AD), each of which has a probability of $25\% \times 25\% = 6.25\%$.

Figure 4. Second movement of prices.

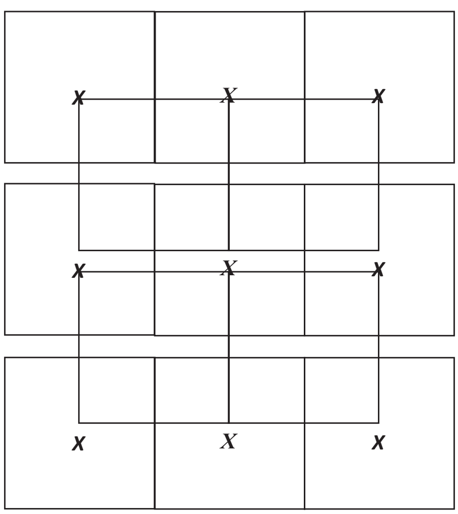


Source: Own elaboration

In this structure, some trajectories lead to the same node. For example, trajectory (1,1) ⇒ (u, A) ⇒ (u, B) and (1,1) ⇒ (u, B) ⇒ (u, A) leads to the same node (u^2, AB) after the second movement, such that the probability of reaching this node is 6.25% + 6.25% = 12.5%. Thus, as shown in Fig. 4, four nodes reach the central node of the pyramid if AD = BC. As a result, the total probability of each node is 25%.

According to the same procedure, if we start from the nine different nodes corresponding to the second price movement, which is represented by X, then Figure 5 presents the possible nodes at the end of the third price movement.

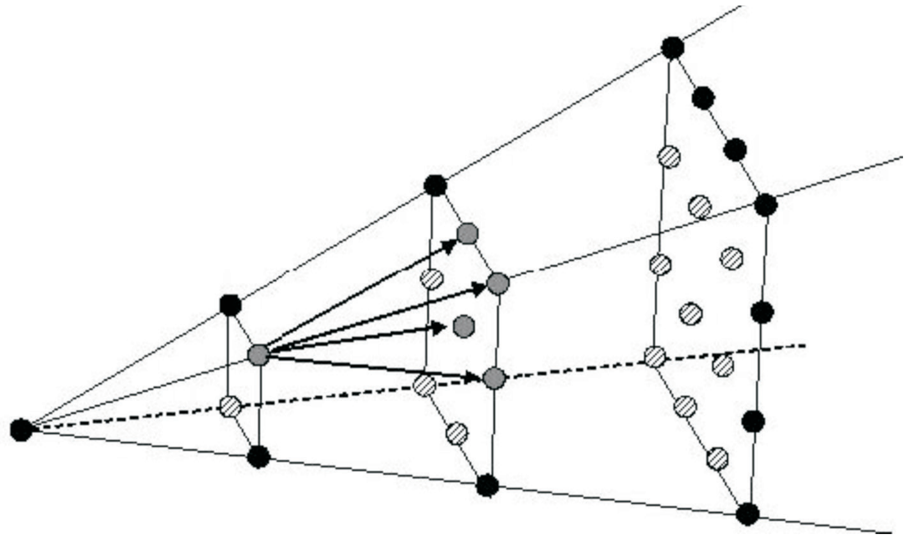
Figure 5. Third movement of prices.



Source: Own elaboration.

If n price movements are contemplated, then the total number of nodes will be $(1 + n)^2$. Therefore, after combining the trajectories of the price movements in Figure 3 to 5, we can represent all of the possible price movements for both underlying assets. As shown in Figure 6, this representation is a horizontal cut of a squared pyramid, with (1,1) at the top and the last price movement at the bottom.

Figure 6. Representation of the binomial pyramid.



Source: Own elaboration

If point (1,1) is the origin, point (x,y) is the final destination, ρ is the correlation coefficient between the returns of both assets, $(\mu_1 h, \mu_2 h)$ is the average of the logarithmic returns, $(\sigma_1\sqrt{h}, \sigma_2\sqrt{h})$ is the standard deviation, t is the time until the maturity date of the option, n is the number of movements, and $h = t/n$ is the amount of time it takes for a price to move between each node in the pyramid, the multiplicative binomial bivariate movement shown in Fig. 7 can be obtained[5].

Figure 7. Third movement of prices.

$(\exp\{\mu_1 h - \sigma_1\sqrt{h}\},$
$\exp\{\mu_2 h - \sigma_2\sqrt{h}[\rho - \sqrt{1-\rho^2}]\})$

$(\exp\{\mu_1 h - \sigma_1\sqrt{h}\},$
$\exp\{\mu_2 h - \sigma_2\sqrt{h}[\rho - \sqrt{1-\rho^2}]\})$

25%	25%
(1.1)	
25%	25%

$(\exp\{\mu_1 h - \sigma_1\sqrt{h}\},$
$\exp\{\mu_2 h - \sigma_2\sqrt{h}[\rho - \sqrt{1-\rho^2}]\})$

$(\exp\{\mu_1 h - \sigma_1\sqrt{h}\},$
$\exp\{\mu_2 h - \sigma_2\sqrt{h}[\rho - \sqrt{1-\rho^2}]\})$

Source: Own elaboration

[5] For further details see Rubinstein (1991).

Additionally, if the risk-free world is represented in the one-dimensional binomial model, then the average of the logarithmic returns can be calculated according to equations (30) and (31):

$$\mu_1 = (\log r / d_1) - \frac{1}{2}\sigma_1^2 \tag{30}$$

$$\mu_2 = (\log r / d_2) - \frac{1}{2}\sigma_2^2 \tag{31}$$

where d_1 and d_2 are defined as the yearly dividends rate of assets 1 and 2, respectively, and r is the risk-free interest rate.

Based on the current values of assets 1 and 2, the risk-free interest rate (r), the standard deviation of the return on asset 1 (σ_1), the standard deviation of the return on asset (σ_2), the correlation between assets 1 and 2 (ρ), and the number of movements (n), the following is derived:

$$u = \exp(\mu_1 h + \sigma_1 \sqrt{h}) \tag{32}$$

$$d = \exp(\mu_1 h - \sigma_1 \sqrt{h}) \tag{33}$$

$$A = \exp\{\mu_2 h + \sigma_2 \sqrt{h}[\rho + \sqrt{(1-\rho^2)})) \tag{34}$$

$$B = \exp\{\mu_2 h + \sigma_2 \sqrt{h}[\rho - \sqrt{(1-\rho^2)})) \tag{35}$$

$$C = \exp\{\mu_2 h - \sigma_2 \sqrt{h}[\rho - \sqrt{(1-\rho^2)})) \tag{36}$$

$$D = \exp\{\mu_2 h - \sigma_2 \sqrt{h}[\rho + \sqrt{(1-\rho^2)})) \tag{37}$$

Thus, the definitions of (u, d) and (A, B, C, D) can be used to construct the movements on the binomial pyramid. Starting from the bottom, the option's value is recursively obtained by discounting four intermediate prices in one in each movement until the top is reached.

4.2. Description of Brandimarte's pyramidal lattice model

Although Rubinstein's (1994) model considers the correlation between the underlying assets by using the possible returns for each asset, Brandimarte's (2006) model incorporates the correlation between the underlying assets by using the probabilities for each price movement, as described below:

If continuous dividend rates of q_1 and q_2 are distributed through assets S_1 and S_2, the price trajectory of each reference asset may be represented by the stochastic processes presented in equations (20) and (21), which are reproduced here:

$$S_1 dt + \sigma_1 S_1 dZ_1 \tag{38}$$

$$dS_2 = (r - q_2) S_2 dt + \sigma_2 S_2 dZ_2 \tag{39}$$

where dZ1 and dZ2 are Wiener processes with an instantaneous correlation $\rho_{1,2}$:

$$dZ_1 dZ_2 = \rho_{1,2} dt \tag{40}$$

Thus, for $x_i = \log S_i$ (i = 1, 2), the following stochastic processes are given:

$$dX_1 = v_1 S_1 dt + \sigma_1 dZ_1 \tag{41}$$

$$dX_2 = v_2 S_2 dt + \sigma_2 dZ_2 \tag{42}$$

where $v_i = r - q_i - \frac{\sigma_i^2}{2}$, i = 1, 2. (43)

As in a typical binomial tree, Brandimarte's (2006) model considers both underlying assets as capable of moving up or down in a fixed amount δx_i in terms of a price logarithm. Similar to the model of Rubinstein (1994), Brandimarte's (2006) model shows that there may be four movements from each node on the tree. Thus, for each movement, four distinct probabilities are defined: p_{uu} (probability of moving up in both underlying assets), p_{ud} (probability of moving up in the first asset and moving down in the second), p_{du} (probability of moving down in the first and up in the second) and p_{dd} (probability of moving down in both underlying assets).

The tree derived, from the probabilities of each prices movement, is evaluated using the output of the two first moments implicit in equations (41) and (42), and the covariance between the two underlying assets. Thus, δx_1 and δx_2 are defined, where δt represents a discrete time interval:

$$\delta x_1 = \sigma_1 \sqrt{\delta t} \tag{44}$$

$$\delta x_2 = \sigma_2 \sqrt{\delta t} \tag{45}$$

In addition, the following system is obtained, with four equations and four variables:

$$p_{uu} + p_{ud} - p_{du} - p_{dd} = \frac{v_1 \sqrt{\delta t}}{\sigma_1} \tag{46}$$

$$p_{uu} - p_{ud} + p_{du} - p_{dd} = \frac{v_2 \sqrt{\delta t}}{\sigma_2} \tag{47}$$

$$p_{uu} - p_{ud} - p_{du} + p_{dd} = \rho_{1,2} \tag{48}$$

$$p_{uu} + p_{ud} + p_{du} + p_{dd} = 1 \tag{49}$$

The equations that define the probabilities of each price movement can be obtained by solving equations (46) to (49).

$$p_{uu} = \frac{1}{4}\left\{1 + \sqrt{\delta t}\left(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right) + \rho_{1,2}\right\} \tag{50}$$

$$p_{ud} = \frac{1}{4}\left\{1 + \sqrt{\delta t}\left(\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\right) - \rho_{1,2}\right\} \tag{51}$$

$$p_{du} = \frac{1}{4}\left\{1 + \sqrt{\delta t}\left(-\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right) - \rho_{1,2}\right\} \tag{52}$$

$$p_{dd} = \frac{1}{4}\left\{1 + \sqrt{\delta t}\left(-\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\right) + \rho_{1,2}\right\} \tag{53}$$

The probabilities described in equations (50) to (53) can be intuitively understood. The probability of moving up for both underlying assets p_{uu} increases as the averages μ_1 and μ_2 increase in relation to their volatilities and as the correlation between the assets increases. However, with respect to the positive movement probability in S_1 and the negative movement probability in S_2, the average μ_2 appears with a negative sign (the higher the value of μ_2, the less probable a negative movement becomes), and the negative correlation makes this joint movement more likely. A similar thought process can be applied to p_{du}; p_{dd} is lower when μ_1 and μ_2 are higher but is higher when the correlation is positive.

Because the tree deals with prices and with their logarithms, the price movements are given by the following:

$$u_i = e^{\delta x_i} = e^{\sigma_i \sqrt{\delta t}} \tag{54}$$

and

$$d_i = \frac{1}{u_i},\ i = 1, 2. \tag{55}$$

4.3. Description of the least-squares Monte Carlo (LSMC) method

The Least-Squares Monte Carlo (LSMC) Method was originally developed by Longstaff and Schwartz (2001) to evaluate vanilla American-style options and Bermuda American options and swaps. With respect to the evaluation of vanilla American options, the exercise strategy is the same as that presented by the lattice models. Based on Monte

Carlo simulations, the expected option cash flow in the risk-free world is estimated by examining the optimum strategy that a holder adopts for the following option:

- The holder desires to maximize the value of the option at any time and considers the early exercise whenever the value obtained in this case is higher than the estimated value of the live option.

In this case, in addition to the exercise value of each moment of time and the simulated price of the underlying asset, knowing the estimated value of the option being kept alive is also required to make the ideal decision. Longstaff and Schwartz's most important contribution was to realize that the value of the option being kept alive could be estimated by utilizing information obtained from the ordinary Least-Squares method. Thus, the application of the ordinary Least-Squares method defines a specific function, referred to as the conditional expectation function, through which the value of the option being kept alive is estimated. A conditional expectation function is defined for each instance of time between the maturity date and the date of issuance[6].

After the cash flow matrix was identified, as generated from the optimal decisions made at each instance of time, where each line represent a price trajectory and each column an instance of time. Then, the value of the option was obtained by discounting all cash flows to present value and calculating the average of these results.

4.3.1. Adaptation of the method to the evaluation of exchange and spread options

To adapt the LSMC method to the evaluation of exchange and spread options, the prices of the underlying assets were conjointly simulated, as shown in the model presented in item 3. Unlike Longstaff and Schwartz's (2001) original model, where only a single asset price was simulated, two matrixes that simulated the price trajectories of both assets 1 and 2 were generated.

Based on these matrixes, the value of the option at the maturity date (i.e., its payoff value) was first obtained by conjointly simulating the prices of the underlying assets, as shown by equations (18) and (19). Thereafter, then starts the recursive evaluation of the derivative from the date prior to maturity T-1, towards the date of issue.

At T-1, if the option is in-the-money, the holder must decide whether to exercise it or to keep it alive. To make the optimum decision, the holder must compare its estimated value with the exercise value. Thus, the vector with two independent variables, $X1_{T-1}$ and $X2_{T-1}$, from the first regression performed in the model represents the values of the underlying assets at T-1 for each simulated trajectory. We consider only the cases in which the derivative is found in-the-money at this date.

The option's values on the maturity date correspond to these trajectories. These values are then discounted up to the date prior to maturity, T-1, at the risk-free interest rate. These discounted amounts represent the dependent variable, Y_{T-1}, from the first regression performed in the model, and the result will be the conditional expectation function for the moment of time T-1, the objective of which is to estimate the continuation value of the option at the given moment. The value of the option at the date before maturity will then be obtained from the following procedure: after replacing each pair of prices of the underlying assets that is jointly simulated and observed in T-1 in the conditional expectation function, the result was compared with the exercise value of the option at the given moment. Based on the holder's previously adopted strategy, the value of the derivative will be equal to the exercise value if the continuation value is

[6] In their article, Longstaff and Schwartz (2001) used the ordinary Least-Squares method to estimate the conditional expectation function. For the evaluation of the vanilla American options, the dependent variable Y is retrograded by considering two constants, X and X^2. However, according to later researchers, other methods such as the weighted Least-Squares method and the generalized Least-Squares method can be more efficient in some cases. In the model developed in the present study, the dependent variable Y is retrograded by considering the constants X1, $X1^2$, X2 and $X2^2$. Nonetheless, it was verified that the results generated presented no relevant sensitivity when the other base-functions were used.

lower than the exercise value or zero.

The derivative evaluation follows recursively. The next step is to check the optimum exercise at the moment of time T-2 using a similar procedure to that applied at T-1. After observing the matrixes containing the trajectories of both underlying asset prices, we represent the underlying asset values in T-2 as the vector with both explicative variables ($X_{1,T-2}$ and $X_{2,T-2}$) from the second regression. However, only the cases in which the derivative is found in-the-money on this date are considered. The variable Y_{T-2} will once again be defined as the discounted value of future cash flows corresponding to these trajectories, which follow moment T-2, disregarded the possibility of exercising the option in T-2. Thus, because the option can only be exercised once, the future cash flows corresponding to T-1 should be discounted for one period. Those options received in T-2 should be discounted for two periods.

After replacing each pair of prices observed in T-2, the next step is to compare the results with the exercise value corresponding to T-2, as was performed for T-1. This procedure must be performed recursively until the moment shortly after the issue date. After identifying the cash flow matrix generated from the optimum decisions made at each moment of time, the value of the option will then be calculated after discounting all present cash flows and obtaining the average of all trajectories.

4.4. Application of the models

To assess the models aimed at evaluating American exchange and spread options, this study uses these models to estimate the prices of American-style options similar to the previously studied European-style exchange and spread options while considering the possibility of the early exercise. The additional data used in the application of the three models are concerned with the dividend rate, which exerts a direct influence on the decision to exercise these options early[7]. The dividends rate considered were the following: PETR3 (3% a year or 0.000821% in the period of 0.1 day), PETR4 (3.514% a year. or 0.000959% in the period of 0.1 day), BBDC3 (2.117% a year or 0.000582% in the period of 0.1 day), BBDC4 (2.38% a year or 0.000653% in the period of 0.1 day), VALE3 (1.421% a year or 0.000392% in the period of 0.1 day), VALE5 (1.669% a year or 0.000460% in the period of 0.1 day), PTIP4 (2.682% a year or 0.000735% in the period of 0.1 day), ITAU4 (2.256% a year or 0.000620% in the period of 0.1 day), ACES4 (3.962% a year or 0.001079% in the period of 0.1 day) and USIM5 (3.709% a year or 0.001012% in the period of 0.1 day)[8].

Tables 7 and 8 present the results for exchange and spread options, respectively, after applying Rubinstein's (1994) and Brandimarte's (2006) lattice models as well as Longstaff and Schwartz's (2001) Least-Squares Monte Carlo (LSMC) model[9]. There is a convergence between the prices estimated by all three models, with differences from the second decimal place onwards.

[7] It is worth highlighting that the effect of this variable is almost null both in European-style exchange options and in European-style spread options. For this reason, the dividends rates were not included in the evaluation of these options. Nevertheless, they could be contemplated with no difficulty at all.
[8] Source: www.bloomberg.com
[9] 25.000 Monte Carlo simulations were considered for each price estimate, as it appeared be sufficient for the convergence of results.

Table 7. Evaluation of American Exchange Options.

	PETR3 and PETR4	BBDC3 and BBDC4	VALE3 and VALE5	PETR4 and PTIP4	ITAU4 and BBDC4	USIM5 and ACES4
Lattice (Rubinstein)	8.660	1.020	13.850	28.989	38.527	41.875
Lattice (Brandimarte)	8.660	1.020	13.850	28.990	38.520	41.900
LSMC	8.668	1.022	13.876	29.011	38.542	41.912

Source: Own elaboration

Table 8. Evaluation of American Spread Options.

	PETR3 and PETR4	BBDC3 and BBDC4	VALE3 and VALE5	PETR4 and PTIP4	ITAU4 and BBDC4	USIM5 and ACES4
Lattice (Rubinstein)	0.329	0.346	0.674	0.906	0.970	2.540
Lattice (Brandimarte)	0.328	0.346	0.674	0.906	0.970	2.540
LSMC	0.329	0.345	0.674	0.894	0.968	2.534

Source: Own elaboration

4.4.1. Sensitivity analysis

The price behaviors of American-style options, in relation to changes in parameters of the model, can be analyzed in the same manner that the European-style exchange options and the European-style spread options were analyzed here. Table 9 presents the analysis of the price behaviors for American-style exchange options of the common stocks and the preferred capital stocks of the "Vale" company (codes VALE3 and VALE5, respectively); the data reveals a 30% reduction in the correlation between the assets (from 0.99 to 0.69), an increase in the annual dividends rate of VALE3 (from 1.421% to 3%) and a 50% increase in the daily volatility of VALE3 (from 2.98% to 4.47%).

As expected, the reduction in the correlation between the underlying assets positively affects the price of the option. The same behavior was observed in relation to the increase in the volatility of the underlying assets. The effect of this change on VALE3 is higher than that observed for VALE5 primarily because of the initial value of the former stock in relation to the latter. However, the increase in the dividends rate of each underlying asset produces distinctive effects. A higher dividends rate represents a reduction in the expected growth rate of the asset. Because the VALE3 appears with a positive sign in the option payoff, an increase in its dividends rate would generate a reduction in its option price. However, an increase in the dividends rate for VALE5 would generate an increase in the option price because the VALE5 appears with negative sign in the option payoff.

Notably, Rubinstein's (1994) and Brandimarte's (2006) models presented low sensitivity to the changes in the variables, but Longstaff and Schwartz's (2001) Least-Squares Monte Carlo (LSMC) model responded more strongly in almost all of the cases studied.

Table 9. Price Sensitivity of American Exchange Options (VALE3 and VALE5)

	Lattice (Rubinstein)	Lattice (Brandimarte)	LSMC
Base Scenario	13.850	13.850	13.876
Reduction of 30% on the correlation	13.856	13.855	13.945
Increase in the dividends rate of VALE3 to 3%	13.850	13.823	13.869
Increase in the dividends rate of VALE5 to 3%	13.869	13.869	13.895
Increase of 50% in the volatility of VALE3	13.850	13.850	13.932
Increase of 50% in the volatility of VALE5	13.850	13.850	13.883

Source: Own elaboration

With respect to the American spread options between VALE3 and VALE5 stocks, Table 10 presents the results of the same sensitivity analysis previously performed for American exchange options between these stocks. In addition, the price sensitivity of the spread option in relation to a 30% reduction of its exercise price was also evaluated.

As expected, the reduction in the correlation between the underlying assets also positively affected the price of the spread option. However, the sensitivity was stronger than that which occurred for the exchange option. In relation to the volatility of the underlying assets, the effect of the changes on the volatility of VALE3 was significantly higher than for the effects of the change on the volatility of VALE5. Because of the higher initial value of the former in relation to the latter and the presence of the exercise price in the option payoff, the final cash flow of its holder is limited by the possibility that the price of VALE5 will move down in relation to the price of VALE3. The increase in the dividends rate of each underlying asset produces distinctive effects for each asset because of the same reasons previously described in relation to the exchange option. In addition, the value of the option presented strong sensitivity to the 30% reduction in the exercise price.

Finally, it is worth highlighting that all three models presented similar results in relation to the changes on the variables.

Table 10. Price Sensitivity of American Spread Options (VALE3 and VALE5)

	Lattice (Rubinstein)	Lattice (Brandimarte)	LSMC
Base Scenario	0.674	0.674	0.674
Reduction of 30% on the correlation	2.132	2.133	2.126
Increase in the dividends rate of VALE3 to 3%	0.660	0.661	0.658
Increase in the dividends rate of VALE5 to 3%	0.684	0.684	0.684
Increase of 50% in the volatility of VALE3	2.062	2.062	2.060
Increase of 50% in the volatility of VALE5	0.788	0.789	0.787
Reduction of 30% in the Exercise Price	4.175	4.175	4.189

Source: Own elaboration

5. Conclusions

Exchange and spread options are complex contracts because they take two underlying assets as the reference points in their payoffs. Nonetheless, these options represent interesting alternatives for those seeking coverage of positions in several assets: stock markets, commodities, interest rates, swaps and futures. The main aim of the present study was to develop adequate evaluation tools for these derivatives and to clarify what is needed for better comprehension and use of these contracts.

In this context, one believes that the models presented here could be useful for market actors who are interested in negotiating these securities, to not only discover fair prices but also analyze the sensitivity of these securities to changes in the parameters.

The model developed here to evaluate European-style spread and exchange options presented good agreement with the results obtained from Margrabe's (1978) and Kirk's (1995) closed-form formulas for exchange options and spread options, respectively. In addition, there was strong agreement among the results obtained by the three models (i.e., the lattice models based on Rubinstein's (1994) and Brandimarte's (2006) three-dimensional binomial pyramids and Longstaff and Schwartz's (2001) adjusted Least-Squares Monte Carlo (LSMC) model) when they were used to evaluate American exchange and spread options.

In relation to possible improvements or extensions of this study, one could select Quasi-Monte Carlo methods, which emerged as an alternative to the traditional Monte Carlo Simulation. In addition, variance reduction techniques, such as the Antithetic Variable technique, Control Variable technique, and Importance Sampling, could be explored.

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Appendix

Kirk (1995) proposed the following formula to evaluate European-style spread call options on two assets.

$$c_K = e^{-r\times T}\left\{F_1 N(d_{K,1}) - (F_2 + K)N(d_{K,2})\right\} \tag{56}$$

$$d_{K,1} = \frac{\ln(F_1/(F_2+K)) + \frac{1}{2}\sigma_K^2 T}{\sigma_K\sqrt{T}} \tag{57}$$

$$d_{K,2} = d_1 - \sigma_K\sqrt{T} \tag{58}$$

$$\sigma_K = \sigma_1^2 - 2\frac{F_2}{F_2+K}\rho\sigma_1\sigma_2 + \left(\frac{F_2}{F_2+K}\right)^2\sigma_2^2 \tag{59}$$

where c_K represents the value of the European-style spread call option with exercise price K, T represents the period until maturity of the option, F_1 and F_2 are the delivery prices for future contracts with maturity equal to the option on two referenced underlying assets, r is the risk-free interest rate, σ_1 and σ_2 are standard deviations of underlying assets logarithmic returns, ρ is the correlation between the logarithmic returns of these assets and $N(\,)$ represents the standard cumulative normal probability distribution function.

In addition, based on the parity between call and put options, the Kirk's approach to evaluate spread put options with exercise price K and maturity at instant T is:

$$p_K = c_K - e^{-r\times T}(F_1 - F_2 - K) \tag{60}$$

Margrabe (1978) proposed the following formula to evaluate exchange options on two assets.

$$c(t) = S_1 N(d_1) - S_2 N(d_2) \tag{61}$$

$$d_1 = \frac{\ln(S_1 / S_2) + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{T-t}} \tag{62}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{63}$$

$$\sigma^2 = \sigma_1^2 - 2\sigma_1\sigma_2\rho + \sigma_2^2 \tag{64}$$

where T represents the period until maturity of the option, σ_1 and σ_2 are standard deviations of underlying assets logarithmic returns, ρ is the correlation between the logarithmic returns of these assets, σ_2 is the volatility of $\ln(S_1 / S_2)$ and $N(\,)$ represents the standard normal cumulative probability distribution function.